

Fuzzy Topological Spaces

Part II (May 17)

“Correct” Fuzzification of Topological Spaces: Functors and the General Tychonoff Theorem

In this second part of his presentation, the speaker will discuss Lowen’s modified definition of a fuzzy topology on a set and its ramifications for the investigation of fuzzy topological spaces. Emphasis will be placed on the use of category theory as a test for a correct generalization of set-based topology and the success in proving a general theorem on products of compact fuzzy topological spaces.

Some Review from Part I

Definitions. (a) A *fuzzy topology* on a set X is a collection δ of fuzzy sets in X satisfying:

(i) $0 \in \delta$ and $1 \in \delta$,

(ii) if μ and ν belong to δ , then so does $\mu \wedge \nu$, and

(iii) if μ_i belongs to δ for each $i \in I$, then so does $\bigvee_{i \in I} \mu_i$.

(b) Let X and Y be fuzzy topological spaces, and let $f : X \rightarrow Y$ be a function. For a fuzzy set ν in Y , the *inverse image* of ν under f is the fuzzy set $f^{-1}(\nu)$ in X defined by the rule

$$f^{-1}(\nu)(x) = \nu(f(x)) = \nu \circ f(x) \text{ for } x \in X,$$

and for a fuzzy set μ in X , the image of μ under f is the fuzzy set $f(\mu)$ in Y defined, for $y \in Y$, by the rule

$$f(\mu)(y) = \begin{cases} \sup\{\mu(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}.$$

(c) Given fuzzy topological spaces (X, δ) and (Y, γ) , a function $f : X \rightarrow Y$ is *fuzzy continuous* if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X ; i.e., if $f^{-1}(\nu) \in \delta$ whenever $\nu \in \gamma$.

Theorems: (a) The fuzzy continuous image of a compact fuzzy topological space is compact.

(b) (Finite Fuzzy Tychonoff Theorem) The product of finitely many compact fuzzy topological spaces is compact. However, there are examples of products of countably many fuzzy topological spaces that fail to be compact.

Example. A non-compact product of countably many compact fuzzy topological spaces.

For each positive integer i let $X_i = \mathbb{N}$, the set of positive integers, let μ_i be the constant fuzzy set in \mathbb{N} given by $\mu_i(x) = \frac{i-1}{i}$ for $x \in \mathbb{N}$, and let $\delta_i = \{0, \mu_i, 1\} \cup \{\mu_i \chi_{\{1,2,\dots,n\}} : n \in \mathbb{N}\}$. Here note that the function $\mu_i \chi_{\{1,2,\dots,n\}}$ is a product; its graph is shown in Figure 2 for the case where $i = 6$ and $n = 10$.

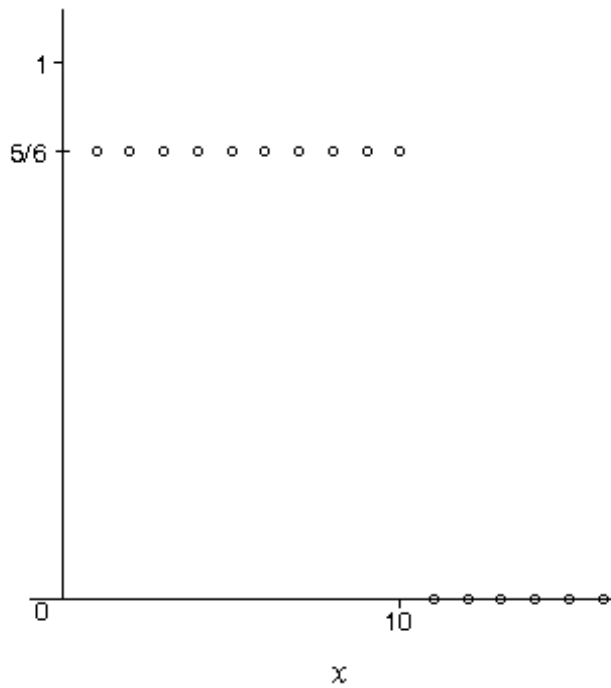


Figure 2. The graph of $\mu_6 \chi_{\{1,2,\dots,10\}}$

δ_i is a fuzzy topology on X_i . Moreover, if $\nu_\alpha \in \delta_i$ ($\alpha \in \Lambda$) and $1 = \bigvee_{\alpha \in \Lambda} \nu_\alpha$, then $\nu_\alpha = 1$ for some α . So (X_i, δ_i) is compact.

Now let $(X, \delta) = (\prod_{i \in \mathbb{N}} X_i, \prod_{i \in \mathbb{N}} \delta_i)$. For $(i, n) \in \mathbb{N} \times \mathbb{N}$,

$$\nu_{i,n} = \pi_i^{-1}(\mu_i \chi_{\{1,2,\dots,n\}}) = \mu_i \chi_{\{1,2,\dots,n\}} \circ \pi_i .$$

is a member of the fuzzy topology $\delta = \prod_{i \in \mathbb{N}} \delta_i$. For a fixed $x = (x_i)_i \in X$,

$$\begin{aligned} \nu_{i,n}(x) &= \mu_i \chi_{\{1,2,\dots,n\}}(x_i) \\ &= \begin{cases} \frac{i-1}{i} & \text{if } x_i \leq n \\ 0 & \text{if } x_i > n. \end{cases} \end{aligned}$$

Given $\varepsilon > 0$, find i with $1 - \varepsilon < \frac{i-1}{i}$. Then for all $n \geq x_i$, $\nu_{i,n}(x) > 1 - \varepsilon$. So $\bigvee_{(i,n) \in \mathbb{N} \times \mathbb{N}} \nu_{i,n}(x) = 1$. But if S is a finite subset of $\mathbb{N} \times \mathbb{N}$, then we can find $N \in \mathbb{N}$ such that if $(i, n) \in S$ then $n < N$. It follows that for $x = (N, N, N, \dots)$ we have $\nu_{i,n}(x) = 0$ for all $(i, n) \in S$, and certainly $\bigvee_{(i,n) \in S} \nu_{i,n} < 1$. Thus, we conclude that (X, δ) is not compact.

An Enlightening General Context

Let L be a complete, complemented, distributive lattice with *zero* 0 and *infinity* 1.

Examples:

- $L = \{0, 1\}$
- $L = [0, 1]$
- $L = \{0, 1\}^2$ with the product order $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$
- $L = [0, 1]^2$ with the product order $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$

An L -fuzzy set in X is a function $\mu : X \rightarrow L$.

An L -fuzzy topological space is a fuzzy topological space with open L -fuzzy sets instead of fuzzy sets.

The theory of topological spaces corresponds to taking $L = \{0, 1\}$.

The theory of fuzzy topological spaces corresponds to taking $L = [0, 1]$.

Definition. Given a cardinal number κ , the infinity 1 of L is said to be κ -isolated if whenever $a_i \in L$ for each index i in an index set I with $|I| \leq \kappa$ and $a_i < 1$ for all $i \in I$, then $\bigvee_{i \in I} a_i < 1$.

Note. For $L = \{0, 1\}$ 1 is κ -isolated for every cardinal κ , and for $L = [0, 1]$ 1 is κ -isolated if and only if κ is a positive integer.

L-Fuzzy Tychonoff Theorem (Goguen, 1973). *Let L be a complete, complemented, distributive lattice with 0 and 1, and let κ be a cardinal number. Then every product of κ compact L -fuzzy topological spaces is compact if and only if 1 is κ -isolated in L .*

Corollaries. (a) ($L = \{0, 1\}$) The usual topological Tychonoff theorem.

(b) ($L = [0, 1]$) The finite Tychonoff theorem for fuzzy topological spaces and guarantee of countably infinite factor counter-examples.

(c) ($L = \{0, 1\}^2$ or $L = [0, 1]^2$) In either case the infinity $(1, 1)$ is 1-isolated but not 2-isolated. So counter-examples exist showing that the product of two compact L -fuzzy topological spaces need not be compact.

First Obstacle: Some products of countably many compact fuzzy topological spaces fail to be compact.

Another Obstacle: Some constant functions from one fuzzy topological space to another fail to be continuous:

Proposition. *Let (X, δ) be a fuzzy topological space. Then every constant function from (X, δ) into another fuzzy topological space is fuzzy continuous if and only if δ contains all constant fuzzy sets in X .*

Definition (Lowen, 1976). (a) A *fuzzy topology* on a set X is a collection δ of fuzzy sets in X that is closed under arbitrary suprema, closed under finite infima, and contains all constant fuzzy sets in X .

(b) A *quasi-fuzzy topology* on a set X is a collection δ of fuzzy sets in X satisfying Chang's definition of fuzzy topology.

Note: In this new context, all fuzzy topologies are quasi-fuzzy topologies. So the previously developed theory for quasi-fuzzy topological spaces (previously Chang's fuzzy topological spaces) holds for our new class of fuzzy topological spaces as well.

However ...

Now ordinary topological spaces are not fuzzy topological spaces.

Definition. Let \mathbf{TOP} denote the category of topological spaces, with continuous functions as morphisms, and let \mathbf{FTOP} denote the category of fuzzy topological spaces, with fuzzy continuous functions as morphisms.

Recall: Associated with a given topological space (X, τ) , there is a natural fuzzy topology $\omega(\tau)$ on X consisting of the l.s.c. fuzzy sets in X . (Since $\omega(\tau)$ contains all constant fuzzy sets in X it is indeed a fuzzy topology in our new sense.)

Definitions. (a) A fuzzy topology δ on X is *topologically generated* if it equals $\omega(\tau)$ for some topology τ on X .

(b) Given a fuzzy topological space (X, δ) , $\iota(\delta)$ denotes the smallest topology on X making all the members of δ l.s.c.

For a topology τ on X , $\iota(\omega(\tau)) = \tau$. However, for a fuzzy topology δ on X , $\omega(\iota(\delta))$ may not equal δ ; it is in fact the smallest topologically generated fuzzy topology containing δ .

Note: (a) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a continuous function of topological spaces, then $f : (X, \omega(\tau)) \rightarrow (Y, \omega(\sigma))$ is fuzzy continuous.

(b) If $f : (X, \delta) \rightarrow (Y, \gamma)$ is a fuzzy continuous function of fuzzy topological spaces, then $f : (X, \iota(\delta)) \rightarrow (Y, \iota(\gamma))$ is continuous.

(c) If δ is topologically generated, so that $\delta = \omega(\tau)$ for some topology τ on X , then any fuzzy continuous function mapping (X, δ) into any fuzzy topological space (Y, γ) is a continuous function from (X, τ) into $(Y, \iota(\gamma))$.

Summarizing in terms of the categories **TOP** and **FTOP**:

Proposition (Lowen, 1976). (a) *If X is a set, then ω is an isotone injection from the set of topologies on X into the set of fuzzy topologies on X .*

(b) *If X is a set, then ι is an isotone surjection from the set of all fuzzy topologies on X onto the set of all topologies on X .*

(c) $\tilde{\omega} : \mathbf{TOP} \rightarrow \mathbf{FTOP}$ defined by $\tilde{\omega}(X, \tau) = (X, \omega(\tau))$ and $\tilde{\omega}(f) = f$ is a covariant functor.

(d) $\tilde{\iota} : \mathbf{FTOP} \rightarrow \mathbf{TOP}$ defined by $\tilde{\iota}(X, \delta) = (X, \iota(\delta))$ and $\tilde{\iota}(f) = f$ is a covariant functor.

(e) $\tilde{\omega}(\mathbf{TOP})$ is a full subcategory of **FTOP**.

Chang's definition of compactness does makes sense in the new context but is not of much use:

No fuzzy topological space can be compact!

Definition (Lowen, 1976). Let (X, δ) be a fuzzy topological space.

(a) A fuzzy set ν in X is *fuzzy compact* if whenever $\bigvee_{i \in I} \mu_i \geq \nu$, where $\mu_i \in \delta$ for all $i \in I$, and $\varepsilon > 0$, then there are finitely many μ_i 's, say $\mu_{i_1}, \dots, \mu_{i_n}$, such that $\bigvee_{j=1}^n \mu_{i_j} \geq \nu - \varepsilon$.

(b) (X, δ) is *fuzzy compact* if and only if each constant fuzzy set in X is fuzzy compact.

(c) (X, δ) is *weakly fuzzy compact* if and only if the constant fuzzy set 1 in X is fuzzy compact.

Theorem. *The following are equivalent for a topological space (X, τ) .*

- (a) (X, τ) is compact.
- (b) $\tilde{\omega}(X, \tau) = (X, \omega(\tau))$ is fuzzy compact.
- (c) $\tilde{\omega}(X, \tau) = (X, \omega(\tau))$ is weakly fuzzy compact.

Proof. (a) \Rightarrow (b). Suppose that (X, τ) is compact. Let α be a constant, with $0 < \alpha \leq 1$ and assume that $\bigvee_{i \in I} \mu_i \geq \alpha$, where $\mu_i \in \omega(\tau)$ for $i \in I$.

Let $\varepsilon > 0$ be given. Without loss of generality, we may assume that $\varepsilon < \alpha$. For each $i \in I$ $U_i = \{(x, t) \in X \times \mathbb{R} : \mu_i(x) > t - \varepsilon/2\}$ is an open subset of $X \times \mathbb{R}$ since μ_i is l.s.c., and also $\bigcup_{i \in I} U_i \supseteq X \times [0, \alpha]$, which is compact.

Thus, there are finitely many indices i_1, \dots, i_n such that $\bigcup_{j=1}^n U_{i_j} \supseteq X \times [0, \alpha]$.

If $x \in X$, then $(x, \alpha - \varepsilon/2) \in X \times [0, \alpha]$ so that for some $j = 1, \dots, n$ we have $(x, \alpha - \varepsilon/2) \in U_{i_j}$ and $\mu_{i_j}(x) > (\alpha - \varepsilon/2) - \varepsilon/2 = \alpha - \varepsilon$. Therefore,

$\bigvee_{j=1}^n \mu_{i_j} \geq \alpha - \varepsilon$. So α is a compact fuzzy set in $(X, \omega(\tau))$.

(b) \Rightarrow (c). Trivial.

(c) \Rightarrow (a). Suppose that $(X, \omega(\tau))$ is weakly fuzzy compact. If $U_i \in \tau$ for each $i \in I$ and $\bigcup_{i \in I} U_i = X$, then $\bigvee_{i \in I} \chi_{U_i} = 1$ and each χ_{U_i} is l.s.c. (since it is the characteristic function of an open set) and hence a member of $\omega(\tau)$. From the weak fuzzy compactness of $(X, \omega(\tau))$, we conclude that there are finitely many indices i_1, \dots, i_n such that $\bigvee_{j=1}^n \chi_{U_{i_j}} \geq 1/2$ which forces $\bigvee_{j=1}^n \chi_{U_{i_j}} = 1$.

Therefore, $\bigcup_{j=1}^n U_{i_j} = X$. So (X, τ) is compact.

Preservation of fuzzy compactness by fuzzy continuous surjections:

Theorem (Lowen, 1976). *Let $f : (X, \delta) \rightarrow (Y, \gamma)$ be a fuzzy continuous function of fuzzy topological spaces.*

(a) *If μ is a fuzzy compact fuzzy set in (X, δ) , then $f(\mu)$ is a fuzzy compact fuzzy set in (Y, γ) .*

(b) *If f is a surjection and (X, δ) is fuzzy compact, then (Y, γ) is fuzzy compact.*

(c) *If f is a surjection and (X, δ) is weakly fuzzy compact, then (Y, γ) is weakly fuzzy compact.*

Lemma. *Let \mathcal{S} be a subbase for a fuzzy topological space (X, δ) .*

(a) *(X, δ) is fuzzy compact if and only if whenever α is a constant fuzzy set in X , $\bigvee_{i \in I} \mu_i \geq \alpha$ where $\mu_i \in \mathcal{S}$ for $i \in I$, and $\varepsilon > 0$ then there are finitely many indices $i_1, \dots, i_n \in I$ such that $\bigvee_{j=1}^n \mu_{i_j} \geq \alpha - \varepsilon$.*

(b) *(X, δ) is weakly fuzzy compact if and only if whenever $\bigvee_{i \in I} \mu_i \geq 1$ where $\mu_i \in \mathcal{S}$ for $i \in I$, and $\varepsilon > 0$ then there are finitely many indices $i_1, \dots, i_n \in I$ such that $\bigvee_{j=1}^n \mu_{i_j} \geq 1 - \varepsilon$.*

Tychonoff Theorem for Fuzzy Compactness (Lowen, 1977). For each $i \in I$ let (X_i, δ_i) be a fuzzy compact fuzzy topological space. Then $(X, \delta) = (\prod_{i \in I} X_i, \prod_{i \in I} \delta_i)$ is fuzzy compact.

Proof. If $0 < \varepsilon < \alpha$, then we will say that a collection of open fuzzy sets of a fuzzy topological space has ε -FUP for α if none of its finite subcollections covers $\alpha - \varepsilon$ (i.e., if none of its finite subcollections has supremum greater than or equal to $\alpha - \varepsilon$).

Let $\alpha > 0$ be a constant fuzzy set in X . We wish to show that α is fuzzy compact in (X, δ) . Now $\mathcal{S} = \{\pi_i^{-1}(\mu_i) : \mu_i \in \delta_i, i \in I\}$ is a subbase for (X, δ) . Let $0 < \varepsilon < \alpha$ and let \mathcal{C} be a subcollection of \mathcal{S} with ε -FUP for α . By part (a) of the lemma, we will be done if we can show that \mathcal{C} does not cover α .

Let $i \in I$ and define $\mathcal{C}_i = \{\mu \in \delta_i : \pi_i^{-1}(\mu) \in \mathcal{C}\}$. We claim that this collection \mathcal{C}_i of open fuzzy sets in (X_i, δ_i) has $\varepsilon/2$ -FUP for $\alpha - \varepsilon/2$. To see this, let $\mu_{i,1}, \dots, \mu_{i,k} \in \mathcal{C}_i$. Note that $\{\pi_i^{-1}(\mu_{i,j}) : j = 1, \dots, k\}$ is a finite subcollection of \mathcal{C} , whence there exists some point $x = (x_i)_{i \in I} \in X$ such that $\bigvee_{j=1}^k \pi_i^{-1}(\mu_{i,j})(x) < \alpha - \varepsilon$. It then follows that

$$\bigvee_{j=1}^k \mu_{i,j}(x_i) = \bigvee_{j=1}^k \mu_{i,j}(\pi_i(x)) = \bigvee_{j=1}^k \pi_i^{-1}(\mu_{i,j})(x) < \alpha - \varepsilon = (\alpha - \varepsilon/2) - \varepsilon/2,$$

as desired.

Since (X_i, δ_i) is fuzzy compact, the constant fuzzy set $\alpha - \varepsilon/2$ is fuzzy compact in (X_i, δ_i) . So \mathcal{C}_i cannot cover $\alpha - \varepsilon/2$, and we can find some point $y_i \in X_i$ such that $(\bigvee \mathcal{C}_i)(y_i) < \alpha - \varepsilon/2$. Having done this for all $i \in I$ set $y = (y_i)_{i \in I}$. If we set $\mathcal{C}'_i = \{\pi_i^{-1}(\mu) : \mu \in \delta_i\} \cap \mathcal{C}$, then as in the proof of the finite Tychonoff theorem in section 4 we conclude $\mathcal{C} = \bigcup_{i \in I} \mathcal{C}'_i$ and

$(\bigvee \mathcal{C}'_i)(y) = (\bigvee \mathcal{C}_i)(y_i)$ so that

$$(\bigvee \mathcal{C})(y) = \bigvee_{i \in I} (\bigvee \mathcal{C}'_i)(y) = \bigvee_{i \in I} (\bigvee \mathcal{C}_i)(y_i) \leq \alpha - \varepsilon/2 < \alpha ,$$

as desired.

Finite Tychonoff Theorem for Weak Fuzzy Compactness (Lowen, 1977). *Let n be a positive integer and, for each $i = 1, \dots, n$, let (X_i, δ_i) be a weakly fuzzy compact fuzzy topological space. Then $(X, \delta) = (\prod_{i=1}^n X_i, \prod_{i=1}^n \delta_i)$ is weakly fuzzy compact.*

Example: A product of countably many weakly compact fuzzy topological spaces that is not weakly fuzzy compact:

Start with our previous example of a non-compact product of countably many compact fuzzy topological spaces. Enlarge each of those fuzzy topologies δ_i on $X_i = \mathbb{N}$ so that all constant fuzzy sets in X_i are open fuzzy sets. With this change, δ_i becomes

$$\delta_i = \{\alpha : \alpha \text{ is a constant fuzzy set in } \mathbb{N}\} \cup \{\mu_{i,n,\alpha} : 0 \leq \alpha < \frac{i-1}{i}\}$$

where, for $x \in \mathbb{N}$,

$$\mu_{i,n,\alpha}(x) = \begin{cases} \frac{i-1}{i} & \text{if } x \leq n \\ \alpha & \text{if } x > n \end{cases}$$

Each (X_i, δ_i) can now be shown to be a weakly fuzzy compact fuzzy topological space that is *not* fuzzy compact. In the product $(X, \delta) = (\prod_{i \in \mathbb{N}} X_i, \prod_{i \in \mathbb{N}} \delta_i)$

the open fuzzy sets $\pi_i^{-1}(\mu_{i,n,0}) = \mu_{i,n,0} \circ \pi_i$ equal the fuzzy sets $\nu_{i,n}$ considered previously. The remarks given previously confirm that, while $\bigvee_{(i,n) \in \mathbb{N} \times \mathbb{N}} \nu_{i,n,0} =$

1, given any ε with $0 < \varepsilon < 1$ and finite subset S of $\mathbb{N} \times \mathbb{N}$ we can always find an $x \in X$ such that $\bigvee_{(i,n) \in S} \nu_{i,n,0}(x) = 0 < 1 - \varepsilon$.