Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling

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Abstract

We consider a nonlinear elliptic boundary value problem on a planar domain. The exponential type nonlinearity in the boundary condition is one that frequently appears in the modeling of electrochemical systems. For the case of a disk we construct a family of exact solutions that exhibit limiting logarithmic singularities at certain points on the boundary. Based on these solutions we develop two criteria that we believe predict the possible locations of the boundary singularities on quite general domains.

1 Introduction

Let Ω be a bounded, simply connected, smooth domain in $\mathbb{R}^2$. The ultimate goal of the work we describe in this paper is to understand the behaviour of solutions to the nonlinear elliptic boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega , \\ \frac{\partial u}{\partial \mathbf{n}} &= \lambda f(u) + g \text{ on } \partial\Omega , \end{aligned} \tag{1}$$

with $f(u) = e^{u/2} - e^{-u/2}$, and λ some real number. This problem, or slightly more complicated variations thereof, show up quite frequently in connection with modeling of electrochemical systems, consisting of an electrolyte and an adjoining metal surface. The surface may be anodic or cathodic, correponding to a corrosion- or a deposition process, respectively. Models using this type of exponential boundary conditions are associated with the names of Butler and Volmer. For an in-depth discussion of the physical modeling, the significance of λ (and other physical parameters that enter into more complicated variations) we refer the reader to books such as [2] and [3]. We also refer to the introduction of [4], where there is a somewhat shorter discussion of some of these issues.

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In the final section of this paper we apply these same two criteria to general domains (by means of a conformal mapping) and (for the case of two singularities) we provide a simple characterization of the possible locations of the limiting singularities.

One of the reasons we are quite interested in the asymptotic behaviour of the solutions to (1) for λ small but positive, is that we think this behaviour may help explain different kinds of surface instabilities observed in real electrochemical systems. Due to the presence of the singularities, it is also very possible that overdetermined measurements (Cauchy data) from such solutions provide good information about the geometry of an inaccessible corroding surface.

2 Numerical approximation of the solution; qualitative behavior

We begin by numerically solving the boundary value problem of interest, in order to highlight some of the qualitative features of the solutions. We restrict our attention to the unit disk D in $\mathbb{R}^2$ and consider

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} &= \lambda f(u) + g, \quad \text{on } \partial D, \end{aligned} \tag{3}$$

where $f(u) = e^{u/2} - e^{-u/2}$.

We parameterize ∂D as $(\cos(\theta), \sin(\theta))$ for $0 \leq \theta < 2\pi$. For simplicity let us consider solutions to (3) that are even with respect to the $x-$axis, *i.e.*, we take g to be given by a cosine series, and we assume that the suitably smooth, harmonic function u may be expanded as

$$u(r,\theta) = \sum_{j=0}^{\infty} a_j r^j \cos(j\theta)$$

for some choice of coefficients a_j. By inserting this expansion into the boundary condition (3) (note that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$) we obtain

$$\sum_{j=0}^{\infty} j a_j \cos(j\theta) = \lambda f\left(\sum_{j=0}^{\infty} a_j \cos(j\theta)\right) + g(\theta), \tag{4}$$

an equation which should be satisfied identically in θ. From this we may attempt to recover appropriate coefficients a_j.

A natural strategy is to choose a fixed n, truncate the infinite sum at $j = n$ and then project both sides of equation (4) onto the span of $\{\cos(k\theta)\}|_{k=0}^{n}$ by integrating against $\cos(k\theta)$ for $k = 0$ to n. This yields

$$k c_k a_k = \lambda \int_0^{2\pi} f\left(\sum_{j=0}^{n} a_j \cos(j\theta)\right) \cos(k\theta)\, d\theta + c_k b_k \tag{5}$$

is a solution for any real c.

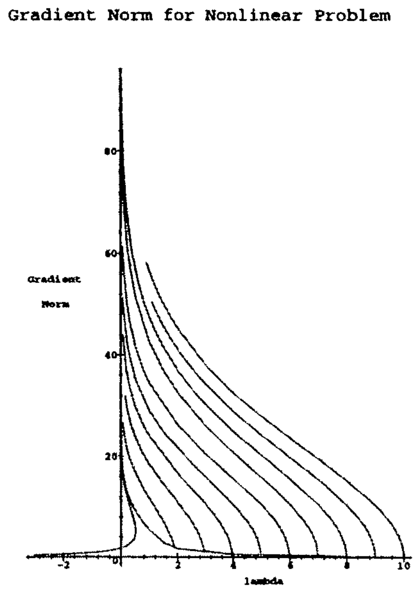

Figure 1

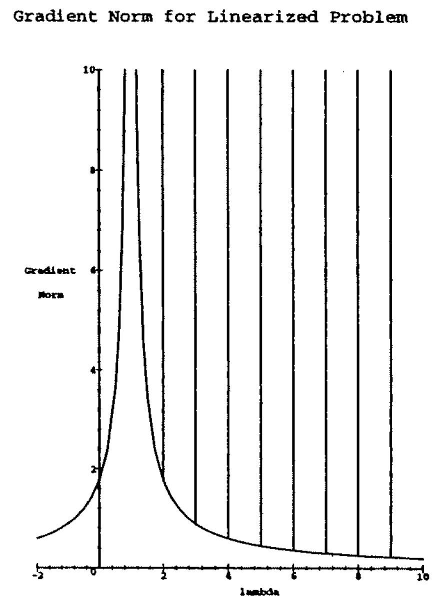

Figure 2

As a specific example consider the case in which $g(\theta) = \cos(\theta)$. One can compute $\|\nabla u\|_2 = \sqrt{\pi}/|1-\lambda|$ for $\lambda < 0$ and for non-integral $\lambda \geq 0$. If $\lambda = 1$ then we have no solution, while if $\lambda = k$ for an integer $k > 1$ then we have infinitely many solutions, and any energy is attainable. For this problem a plot similar to Figure 1 can be constructed. Figure 2 shows $\|\nabla u\|_2$ versus λ for the linearized problem, in which the vertical lines represent the fact that by adding in a suitable multiple of $r^k \cos(k\theta)$ one can obtain any energy. For λ sufficiently far from zero (when u is small) the behavior of the nonlinear problem is qualitatively quite similar to that of the linearized problem. Note that Figures 1 and 2 are quite similar; in Figure 1 the vertical lines arising from the eigenvalues for the linearized problem have merely been distorted.

As mentioned above, as λ approaches zero from the right the solutions along all growing branches develop singularities on ∂D. Figure 3 below shows a solution with boundary data $g(\theta) = \cos(\theta)$ for $\lambda = 1.0 \times 10^{-5}$ on the $n = 2$ branch. Here four singularities develop at $\theta = 0, \pi/2, \pi, 3\pi/2$. We find in general that as $\lambda > 0$ approaches zero the solution on the nth branch develops $2n$ uniformly spaced singularities of alternating sign on ∂D. The singularities appear to be logarithmic, and this behavior does not seem to depend on g. The exact behavior of the solutions in the special case that $g \equiv 0$ is examined in the next section.

In what follows we make use of the identities

$$\begin{aligned} \prod_{k=0,\text{ even}}^{2n-2}(1+\mu^2-2\mu\cos(\theta-\theta_k)) &= \mu^{2n}-2\mu^n\cos(n\theta)+1 \\ \prod_{k=1,\text{ odd}}^{2n-1}(1+\mu^2-2\mu\cos(\theta-\theta_k)) &= \mu^{2n}+2\mu^n\cos(n\theta)+1\ . \end{aligned} \tag{7}$$

These can be proved by noting that in each equation the left and right sides are polynomials in μ of degree $2n$, with the same roots and the same leading coefficients. We will also make use of

$$\begin{aligned} \sum_{k=0,\text{even}}^{2n-2}\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)} &= \frac{n(\mu^{2n}-1)}{(\mu^2-1)(\mu^{2n}-2\mu^n\cos(n\theta)+1)} \\ \sum_{k=1,\text{odd}}^{2n-1}\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)} &= \frac{n(\mu^{2n}-1)}{(\mu^2-1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}\ . \end{aligned} \tag{8}$$

These can be proved by taking the logarithm of both sides of the respective product identities (7), then differentiating with respect to μ and rearranging.

We begin by computing $\frac{\partial v_\lambda}{\partial \mathbf{n}}$ and $\lambda f(v_\lambda)$ explicitly. First,

$$\begin{aligned} \frac{\partial v_\lambda}{\partial x} = &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{x-\mu x_k}{(x-\mu x_k)^2+(y-\mu y_k)^2} \\ &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{x-\mu x_k}{1+\mu^2-2\mu\cos(\theta-\theta_k)} \end{aligned}$$

where $\theta_k = k\pi/n$ and (x,y) lies at angle θ. Here we use the facts that $x^2+y^2 = x_k^2+y_k^2 = 1$ and $xx_k+yy_k = \cos(\theta-\theta_k)$. A similar computation can be made for $\frac{\partial v_\lambda}{\partial y}$ and since $\mathbf{n} = (x,y)$ on the unit circle we obtain

$$\begin{aligned} \frac{\partial v_\lambda}{\partial \mathbf{n}} = x\frac{\partial v_\lambda}{\partial x}+y\frac{\partial v_\lambda}{\partial y} &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{1-\mu\cos(\theta-\theta_k)}{1+\mu^2-2\mu\cos(\theta-\theta_k)} \\ &= 2c\sum_{k=0}^{2n-1}(-1)^k\left(\frac{1}{2}+\frac{1-\mu^2}{2(1+\mu^2-2\mu\cos(\theta-\theta_k))}\right) \\ &= c(1-\mu^2)\sum_{k=0}^{2n-1}(-1)^k\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)}. \end{aligned} \tag{9}$$

We can make use of the identities (8) in equation (9) to find that

$$\frac{\partial v_\lambda}{\partial \mathbf{n}} = \frac{-4cn\mu^n(\mu^{2n}-1)\cos(n\theta)}{(\mu^{2n}-2\mu^n\cos(n\theta)+1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}. \tag{10}$$

Now consider the quantity $\lambda f(v_\lambda)$. Using $f(v) = e^{v/2}-e^{-v/2}$ as well as $(x-\mu x_k)^2+(y-\mu y_k)^2 = 1+\mu^2-2\mu\cos(\theta-\theta_k)$ we find that

$$\begin{aligned} f(v_\lambda) &= \left(\frac{\prod_{k=0,\text{ even}}^{2n-2}(1+\mu^2-2\mu\cos(\theta-\theta_k))}{\prod_{k=1,\text{ odd}}^{2n-1}(1+\mu^2-2\mu\cos(\theta-\theta_k))}\right)^{c/2} \\ &\quad -\left(\frac{\prod_{k=1,\text{ odd}}^{2n-1}(1+\mu^2-2\mu\cos(\theta-\theta_k))}{\prod_{k=0,\text{ even}}^{2n-2}(1+\mu^2-2\mu\cos(\theta-\theta_k))}\right)^{c/2} \end{aligned} \tag{11}$$

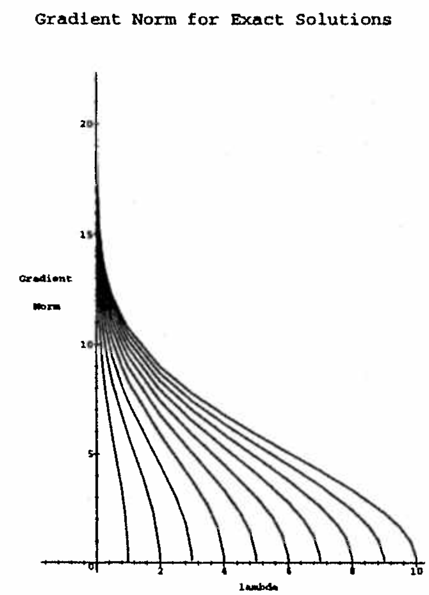

Figure 4

which is quite similar to Figure 1. Note that in the case $g \equiv 0$ we have a solution $v_\lambda \equiv 0$ for all λ, so that the λ axis is the branch which extends to the left half plane.

We can determine the precise asymptotic behavior of the integral in equation (14) as λ approaches zero. Let $N_\epsilon = (\partial D) \cap (\cup_{k=0}^{2n-1} B_\epsilon(p_k))$ (where $B_\epsilon(p)$ is an open disk of radius ϵ around p) denote a neighborhood of the p_k in ∂D and set $\partial D_\epsilon = \partial D \setminus N_\epsilon$, so ∂D_ϵ is a subset of ∂D which excludes a small interval around each p_k. It's easy to see that as λ approaches zero the function v_λ remains uniformly bounded on ∂D_ϵ, and equation (10) makes it clear that $\frac{\partial v_\lambda}{\partial \mathbf{n}}$ approaches zero uniformly on ∂D_ϵ (since the denominator on the right in (10) is bounded away from zero on ∂D_ϵ, and $\mu \to 1$ as $\lambda \to 0^+$). As a consequence, $\int_{\partial D_\epsilon} v_\lambda \frac{\partial v_\lambda}{\partial \mathbf{n}}\, d\sigma_x$ approaches 0 as λ approaches 0 (for any fixed ϵ.)

From this observation and the symmetry of the solution with respect to the $2n$ poles, we see that the asymptotic behavior of the integral on the right in equation (14) will be the same as that of

$$2n \int_{-\epsilon}^{\epsilon} \frac{-16n\mu^n(\mu^{2n}-1)\cos(n\theta)\ln\left(\frac{\mu^{2n}-2\mu^n\cos(n\theta)+1}{\mu^{2n}+2\mu^n\cos(n\theta)+1}\right)}{(\mu^{2n}-2\mu^n\cos(n\theta)+1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}\, d\sigma \ ,$$

for fixed small ϵ, in which the integral above is the contribution of the pole near $(1,0)$. Making the approximations $\mu^n = 1 + O(\lambda)$, $\mu^{2n} + 2\mu^n \cos(n\theta) + 1 =$

and

$$\lambda f(v_\lambda) = \lambda\frac{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))} - \lambda\frac{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} .$$

As a consequence

$$\begin{aligned} \frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda) &= \lambda\frac{2(1-\alpha^2)/\lambda + (1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \\ &\quad -\lambda\frac{2(1-\beta^2)/\lambda + (1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))} . \end{aligned} \tag{17}$$

Consider now the first term of this residual. A necessary and sufficient condition that this first term be uniformly bounded (in θ and λ) is that the numerator of the fraction vanishes at $(\lambda, \theta) = (0, \theta_0)$, *i.e.*,

$$-4\alpha'(0) + 2(1-\cos(\theta_0-\theta_1)) = 0 . \tag{18}$$

To see this, note that the denominator of this fraction is bounded from below by $c\,[(1-\alpha)^2 + (\theta-\theta_0)^2]$. Furthermore note that if (18) is satisfied then $\alpha'(0) \neq 0$, and thus

$$\begin{aligned} |\text{First term of residual}| &\le \lambda\frac{C(\lambda + |\theta-\theta_0|)}{c\,[(1-\alpha)^2+(\theta-\theta_0)^2]} \\ &\le C\frac{\lambda^2+\lambda|\theta-\theta_0|}{(\alpha'(0)\lambda)^2+(\theta-\theta_0)^2} \le C . \end{aligned}$$

Similarly the second term of the residual (17) is uniformly bounded if and only if

$$-4\beta'(0) + 2(1-\cos(\theta_0-\theta_1)) = 0 .$$

In summary, the entire residual $\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$ is uniformly bounded in θ and λ exactly when

$$\alpha'(0) = \beta'(0) = \frac{1}{2}(1-\cos(\theta_0-\theta_1)) .$$

We now proceed further and ask the question: "exactly when is the residual (17) bounded by $C\lambda$ uniformly in θ ?" The first term of (17) is clearly bounded by $C\lambda$ uniformly in θ, exactly when the numerator (in addition to vanishing) has first derivatives with respect to λ and θ that vanish at $(\lambda, \theta) = (0, \theta_0)$. The corresponding constraints are

$$-2[(\alpha'(0))^2 + \alpha''(0)] + 2\beta'(0)(1-\cos(\theta_0-\theta_1)) = 0 ,$$

and

$$2\sin(\theta_0-\theta_1) = 0 .$$

Similarly the second term of (17) is bounded by $C\lambda$ uniformly in θ, exactly when

$$-2[(\beta'(0))^2 + \beta''(0)] + 2\alpha'(0)(1-\cos(\theta_0-\theta_1)) = 0 ,$$

and

$$\lambda \int_{\partial D} F(v_\lambda)\, d\sigma = 4\pi[1-\cos(\theta_0-\theta_1)] \left(\frac{1}{\alpha'(0)} + \frac{1}{\beta'(0)}\right) + o(1) \ , \qquad (21)$$

into (19). We now proceed to verify the identities (20) and (21). To that end

$$\begin{aligned} &\frac{1}{2}\int_{\partial D} \frac{\partial v_\lambda}{\partial \mathbf{n}} v_\lambda \, d\sigma \qquad (22)\\ &= 2\int_0^{2\pi} \left[\frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} - \frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\right] \\ &\quad \times \Big[\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))] \\ &\qquad - \log[(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))]\Big] \, d\theta \ . \end{aligned}$$

Let us first calculate

$$\int_0^{2\pi} \frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]\, d\theta = I + II \ ,$$

where

$$I = \int_{S_1} \frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]\, d\theta \ ,$$

with $S_1 = (0,2\pi) \cap \{1-\cos(\theta-\theta_0) \geq \lambda^{1/4}\}$, and

$$II = \int_{S_2} \frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]\, d\theta \ ,$$

with $S_2 = (0,2\pi) \cap \{1-\cos(\theta-\theta_0) < \lambda^{1/4}\}$. For $\theta \in S_1$ we immediately get

$$\left|\frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}\right| \leq C\lambda^{3/4} \ .$$

Therefore

$$|I| \leq C\lambda^{3/4} \int_0^{2\pi} |\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]|\, d\theta \leq C\lambda^{3/4} \ . \qquad (23)$$

We may without loss of generality suppose that the polar coordinate system has been chosen so that θ_0 and θ_1 are both different from 0 (and 2π). For sufficiently small λ the set $S_2 = (0,2\pi) \cap \{1-\cos(\theta-\theta_0) < \lambda^{1/4}\}$ then splits into two equal parts S_2^+ and S_2^-: one in which $\theta_0 < \theta$ and one in which $\theta < \theta_0$. The contribution to the integral is the same from the two sets. We introduce the new variable of integration

$$s = \frac{\sqrt{2\alpha(1-\cos(\theta-\theta_0))}}{\alpha - 1} \ ,$$

in which form we recognize part of the integral as a double layer potential with density $\log|(x,y)-\beta(x_1,y_1)|^2$ evaluated at the point $\alpha(x_0,y_0)$ (which lies outside D). Since $(x_0,y_0)\neq(x_1,y_1)$ it now follows immediately from the "jump relations" for double layer potentials that the term (26) converges to

$$\int_{\partial D}\left[\frac{2((x,y)-(x_0,y_0))\cdot\mathbf{n}_{(x,y)}}{|(x,y)-(x_0,y_0)|^2}-1\right]\log|(x,y)-(x_1,y_1)|^2\,d\sigma$$
$$-2\pi\log|(x_0,y_0)-(x_1,y_1)|^2\,, \tag{27}$$

as $\lambda\to 0^+$. Since ∂D is the unit circle it is easy to see that

$$\frac{2((x,y)-(x_0,y_0))\cdot\mathbf{n}_{(x,y)}}{|(x,y)-(x_0,y_0)|^2}=1\,,\qquad (x,y)\in\partial D\,.$$

Inserting this into (27) we conclude that the term (26) converges to

$$-2\pi\log|(x_0,y_0)-(x_1,y_1)|^2\,, \tag{28}$$

as $\lambda\to 0^+$. By the exact same argument we get that

$$\int_0^{2\pi}\frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]\,d\theta\,,$$

converges to

$$-2\pi\log|(x_0,y_0)-(x_1,y_1)|^2\,, \tag{29}$$

as $\lambda\to 0^+$. Substituting (24), (25), (28) and (29) into (22) we arrive at

$$\begin{aligned}\frac{1}{2}\int_{\partial D}\frac{\partial v_\lambda}{\partial\mathbf{n}}v_\lambda\,d\sigma &= -16\pi\log\lambda-8\pi\log\alpha'(0)-8\pi\log\beta'(0)\\ &\quad -16\int_0^{+\infty}\frac{1}{s^2+1}\log(s^2+1)\,ds\\ &\quad +8\pi\log|(x_0,y_0)-(x_1,y_1)|^2\;+\;o(1)\quad,\end{aligned}$$

which is exactly the same as (20).

It only remains to verify (21). We calculate

$$\begin{aligned}\lambda\int_{\partial D}F(v_\lambda)\,d\sigma &= 2\lambda\int_{\partial D}(e^{v_\lambda/2}+e^{-v_\lambda/2})\,d\sigma\\ &= 2\lambda\int_0^{2\pi}\frac{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\,d\theta\\ &\quad +2\lambda\int_0^{2\pi}\frac{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}\,d\theta\,.\end{aligned} \tag{30}$$

Upon replacing $1-\beta^2=-2\beta'(0)\lambda+O(\lambda^2)$ by 2λ, and $\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]$ by $(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))$ we may use the exact same procedure that was used to compute the formula (29) for

$$\int_0^{2\pi}\frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]\,d\theta\,,$$

5 Location of singularities on arbitrary domains

Consider now the boundary value problem

$$\begin{aligned} \triangle u &= 0 \ \text{in} \ \Omega \ , \\ \frac{\partial u}{\partial \mathbf{n}} &= \lambda f(u) + g \ \text{ on } \ \partial\Omega \ , \end{aligned} \tag{33}$$

on a smooth, bounded, simply connected domain $\Omega \subset \mathbb{R}^2$. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, by identifying the point (x, y) with the complex number $z = x + iy$. According to Riemann's Mapping Theorem there exists an analytic function $\Phi(\cdot)$, such that the mapping $z \rightarrow \Phi(z)$ maps Ω one to one onto the unit disk, D. From elliptic regularity theory we know that Φ has a smooth extention to $\bar{\Omega}$, and we furthermore know that the extension of $|\frac{d\Phi(z)}{dz}| = |\det[D\Phi(x, y)]|^{1/2}$ does not vanish on $\bar{\Omega}$. The function $w = u \circ \Phi^{-1}$ now satisfies

$$\begin{aligned} \triangle w &= 0 \ \text{in} \ D \ , \\ h\frac{\partial w}{\partial \mathbf{n}} &= \lambda f(w) + g \quad \text{on} \ \ \partial D \ , \end{aligned} \tag{34}$$

where h denotes the (boundary) function $h(\cdot) = |\det[D\Phi(\Phi^{-1}(\cdot)]|^{1/2}$. We may think of h as a function of the angular variable $\theta : h(\theta) = h(e^{i\theta})$.

We believe that singularities will develop in w at specific points on ∂D, as $\lambda > 0$ approaches 0. We also believe that these singularities and their locations will mimic those that develop in one of the non-trivial solutions to the homogeneous boundary value problem

$$\begin{aligned} \triangle v &= 0 \ \text{in} \ D \ , \\ h\frac{\partial v}{\partial \mathbf{n}} &= \lambda f(v) \quad \text{on} \ \ \partial D \ . \end{aligned} \tag{35}$$

In the previous section we identified two criteria that correctly selected the possible locations of the limiting singularities for solutions to (35) in the case when Ω was the unit disk, *i.e.*, when $h(\theta) = 1$. We shall now calculate what these criteria predict concerning the singularity locations in the case when h is not identically 1 (and two singularities develop). Fortunately the two predictions coincide – and we do conjecture that these locations are those which will appear in solutions to (35) (and (34)) in the limit as $\lambda \rightarrow 0^+$ (when two singularities develop). By the conformal mapping Φ^{-1} these locations get carried to the (conjectured) locations for the limiting singularities of solutions to the problem (33). We note that even though we here only consider the case of two singularities, similar calculations could be carried out for any even number of singularities.

With v_λ given by the ansatz (16) we calculate

$$h\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$$

It is now very simple to calculate that the stationarity conditions of the "renormalized energy" $E^*(\theta_0, \theta_1, \alpha'(0), \beta'(0))$ with respect to $\alpha'(0)$, $\beta'(0)$, θ_0 and θ_1 (in that order) amount to exactly the conditions (36) and (38). The equations (38) determine the locations of the limiting singularities; given these locations the equations (36) then determine $\alpha'(0)$ and $\beta'(0)$. Just as we experienced in the last section $\alpha''(0)$ and $\beta''(0)$ are not determined by stationarity of the "renormalized energy".

Observation 3

As a consequence of the calculations carried out above and in the previous section, we conjecture that in the case when two limiting singularities develop in a solution to (33), then these singularities will be located at points $(x_0, y_0) = \Phi^{-1}(\cos(\theta_0), \sin(\theta_0))$ *and* $(x_1, y_1) = \Phi^{-1}(\cos(\theta_1), \sin(\theta_1))$, *where* θ_0 *and* θ_1 *satisfy*

$$\frac{h'(\theta_0)}{h(\theta_0)} = \frac{\sin(\theta_0 - \theta_1)}{1 - \cos(\theta_0 - \theta_1)} = -\frac{h'(\theta_1)}{h(\theta_1)} . \tag{40}$$

We illustrate the assertion of this conjecture with a simple numerical example.

Example

Let Ω_a be the image of the unit disk under the mapping $z \to \exp(az)$ for $0 < a < \pi$. As the mapping $\Phi_a : \Omega_a \to D$ we may thus take $\Phi_a(z) = \frac{1}{a}\log(z)$. Simple calculations give that

$$\left|\frac{d\Phi}{dz}\right| = \frac{1}{a|z|} ,$$

$$h(\theta) = h(\cos\theta + i\sin\theta) = \frac{1}{a\,|\exp(a\cos\theta + ia\sin\theta)|} = \frac{1}{a}\exp(-a\cos\theta) ,$$

and

$$h'(\theta) = \sin\theta\exp(-a\cos\theta) .$$

It follows immediately from (40) that the conjectured limiting singularity locations correspond to angles θ_0 and θ_1, that satisfy

$$a\sin\theta_0 = \frac{\sin(\theta_0 - \theta_1)}{1 - \cos(\theta_0 - \theta_1)} = -a\sin\theta_1 . \tag{41}$$

We note that if (θ_0, θ_1) satisfies these equations, so does (θ_1, θ_0) (in terms of solutions to (35) this just reflects the fact that if v is a solution, so is $-v$). Let us therefore, to eliminate this trivial symmetry, for the moment adopt the convention that $0 \le \theta_0 < \theta_1 < 2\pi$. The equations (41) have the solutions

$$(\theta_0, \theta_1) = (0, \pi) \quad \text{and} \tag{42}$$

$$(\theta_0, \theta_1) \quad \text{where} \quad a\sin\theta_0 = \frac{\sin(2\theta_0)}{1 - \cos(2\theta_0)} \quad \text{and} \quad \theta_1 = 2\pi - \theta_0 . \tag{43}$$

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