Symmetry and Tiling Groups for Genus 4 and 5

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Abstract

All symmetry groups for surfaces of genus 2 and 3 are known. In this paper, we classify symmetry groups and tiling groups with three branch points for surfaces of genus 4 and 5. Also, a class of symmetry groups that are not tiling groups is presented, as well as a class of odd order non-abelian tiling groups.

1 Introduction

Given an orientable compact surface X, we may consider a subgroup G of $\mathrm{Aut}(X)$ the group of conformal automorphisms of X. We are especially interested in the case where G is the subgroup of orientation preserving symmetries generated by the reflections in the edges of the polygons of a tiling of X. A *tiling* is a complete covering of the surface by non-overlapping polygons. In this paper we consider tilings of surfaces with triangles. We are concentrating only on tilings by triangles because such tilings yield a large number of symmetries of the surface, they are the simplest cases to analyze group theoretically, and because, up to topological equivalence, almost all finite group actions on a surface arise from subgroups of triangle tilings. In [Br], Broughton classifies all possible groups of orientation-preserving automorphisms of surfaces of genus 2 and 3. In section 3 of this paper, Broughton's work is continued by classifying possible three branch orientation-preserving symmetry groups of surfaces of genus 4 and 5, and determining which of these correspond to tilings of the surface. These results, as well as the triangle group actions found in [Br], are given in the tables in section 6. In section 4, some propositions are given that deal with specific classes of symmetry and tiling groups.

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2 Symmetry Groups and Tiling Groups

In this paper, we will be considering *kaleidoscopic* tilings of surfaces. In a kaleidoscopic tiling, two tiles sharing an edge are mirror reflections of each other. Given an edge of a tile, the reflection in that edge is a symmetry of the surface fixing that edge. Note that we are assuming that the reflection permutes the tiles. The set of all fixed points of a reflection is called a *mirror*. The tilings considered in this paper also have the property that each mirror is the union of edges of tiles. These ideas are illustrated by the icosahedral tiling of the sphere in Figure 1,where the mirrors are great circles.

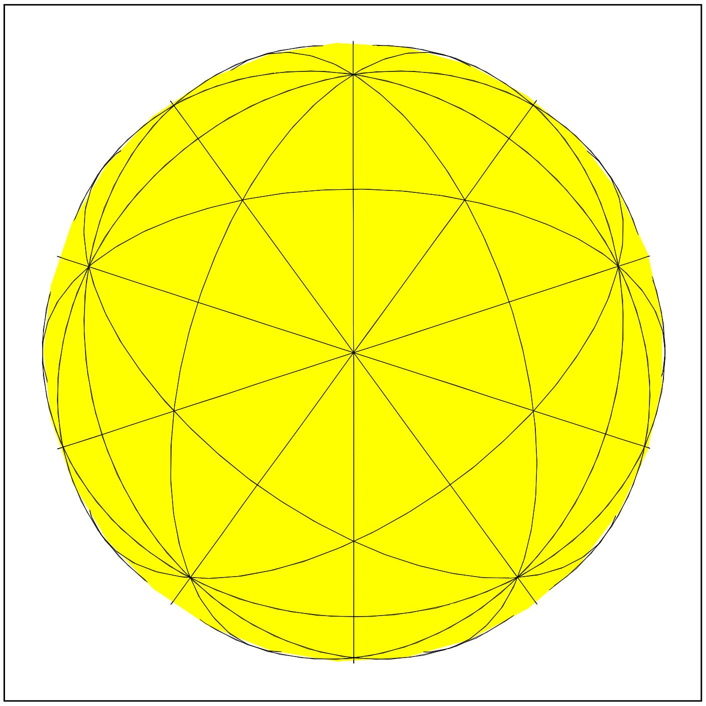

Figure 1. Icosahedral tiling - top view

For the remainder of the paper, the term *tiling group* will mean the group of orientation preserving isometries of X generated by a tiling by triangles. The term *symmetry group* refers to a group of isometries corresponding to three branch points but not necessarily coming from a tiling. It will be specified when the *full tiling group* (including reflections) is being considered. More precise definitions will be given in section 2.

Suppose that we have tiled a surface of genus σ with triangles as described above. A consequence of the conditions on the tiling is that all of the angles that meet at a vertex will be congruent. Since the sum of all of the angles at a vertex must be 2π, we may then determine the measure of the angles in a typical tile by counting the number of tiles that meet at the vertices. Furthermore, because every edge is part of a mirror, there is an even number of angles meeting at each vertex. Therefore, a typical tile is a triangle with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$, where l, m, n are integers ≥ 2. We will call such a triangle an (l, m, n)-triangle. Also note, according to the Gauss-Bonnet theorem, that if $\sigma = 0$, then $\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} > \pi$ (spherical geometry), if $\sigma = 1$, then $\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} = \pi$ (euclidean geometry), and if $\sigma \geq 2$, then we have $\frac{\pi}{l}+\frac{\pi}{m}+\frac{\pi}{n}<\pi$ (hyperbolic geometry).

Now let us choose a tile on the surface and call this the identity tile (see Figure 2). Call the reflections in the sides opposite the angles of measure $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$, r, p, and q, respectively. If we look at these reflections as elements of the group of isometries of the identity tile on the surface, we see that $p^2=q^2=r^2=1$, where 1 represents the identity isometry. If we begin with the identity tile, we should be able to move to any other tile on the surface by repeatedly applying p, q, and r, since X is connected. We will call $G^*=\langle p,q,r\rangle$ the *full tiling group* of the surface with respect to this tiling. Notice that $a=pq$, $b=qr$, and $c=rp$ are rotations of angles $\frac{2\pi}{l}$, $\frac{2\pi}{m}$, and $\frac{2\pi}{n}$, respectively, around the vertices of angle measure half of the corresponding rotation. Call these rotations a, b, and c respectively. We have $a,b,c\in G^*$, and a, b, and c are orientation-preserving isometries of X. We also have

$$a^l=b^m=c^n=1$$

and

$$abc=pqqrrp=1,$$

so that $ab=c^{-1}$, so $\langle a,b,c\rangle=\langle a,b\rangle$. Let us call (a,b,c) an (l,m,n)-*generating triple*, and call $G=\langle a,b\rangle$ the *tiling group* of the surface with respect to this tiling. This is the group of all orientation-preserving isometries of the identity tile onto other tiles on the surface. It turns out that G is an index 2 subgroup of G^*, so that $G\lhd G^*$. Furthermore, we also have the following conjugacy relationships with q:

$$a^q=q^{-1}aq=qaq=a^{-1}$$

$$b^q=q^{-1}bq=qbq=b^{-1}$$

(**Notation:** We use $y^x=x^{-1}yx$ and $[x,y]=x^{-1}y^{-1}xy$). This action of q on G induces an involutory automorphism (order 2) of G, call it θ. In fact, we have $G^*\cong\langle\theta\rangle\ltimes G$, where θ satisfies:

$$\theta(a)=a^{-1},\ \theta(b)=b^{-1},\ \theta^2=id \tag{1}$$

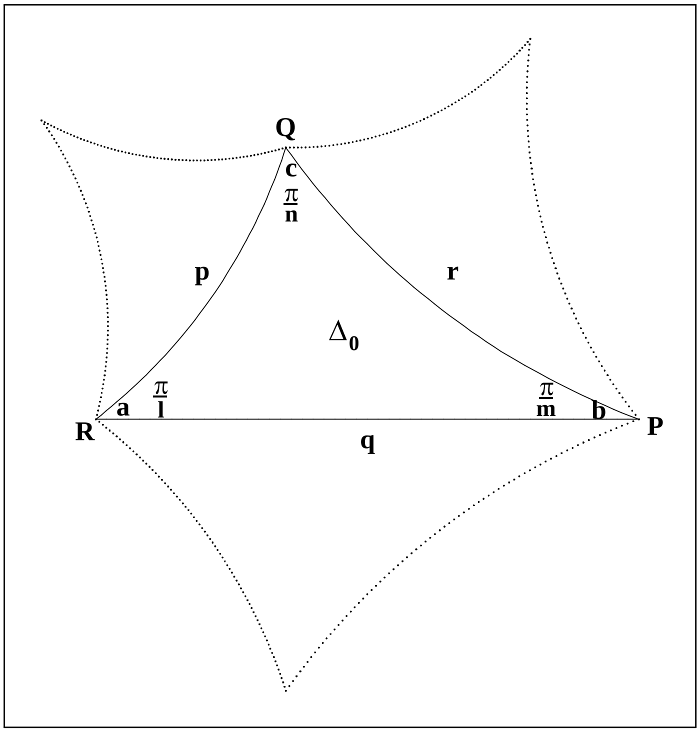

Figure 2. The identity tile Δ_0, reflected images, and the genrators of G^* and G

In the first major work on tilings of surfaces by Hurwitz [Hur], we find the following formula, known as the Riemann-Hurwitz equation, relating the genus of the surface, the order of the tiling group, and the orders of the generating rotations (it may be proved using the Euler characteristic):

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right) \tag{2}$$

So far, we have taken a tiling of a surface and found properties of the tiling group. The following theorem guarantees that a given group is the tiling group of a tiling of a surface.

Theorem 1 *[S] Let G be a group. If (a, b, c) is an (l, m, n)-generating triple for G, the σ obtained from formula (2) is a nonnegative integer, and there is an automorphism θ satisfying (1). Then G is a tiling group of a tiling of a surface of genus σ by (l, m, n) triangles.*

If there is an (l, m, n)-generating triple for G and σ is a nonnegative integer, but there is not necessarily a good θ, then G is a group of orientation-preserving symmetries (or group action) of a surface of genus σ. We will call such a G a *symmetry group* of a surface of genus σ. A class of examples of symmetry groups that are not tiling groups is given in section 4.

Remark 2 *Note that an abelian symmetry group is always a tiling group via the automorphism* $\theta(g) = g^{-1}$ *for all* $g \in G$.

Many times we would like to know when two different (l, m, n)-generating triples correspond to equivalent groups of symmetries. The following theorem is what we need to do this. It follows from Propositions 2.2 and 2.7 of [Br].

Theorem 3 *Let* (a, b, c) *be an* (l, m, n)*-generating triple for* G *corresponding to the surface* X. *Then the* (l, m, n)*-generating triple* (a', b', c') *corresponds to a conformally equivalent surface, commuting with the group actions of* G, *if and only if one of the following holds:*

1. $l < m < n$ *and there is an* $\omega \in \mathrm{Aut}(G)$ *such that*

$$(a', b', c') = \omega \cdot (a, b, c),$$

2. $l = m < n$ *and there is an* $\omega \in \mathrm{Aut}(G)$ *such that at least one of*

$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (aba^{-1}, a, c)$$

holds,

3. $l < m = n$ *and there is an* $\omega \in \mathrm{Aut}(G)$ *such that at least one of*

$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (a, bcb^{-1}, b)$$

holds,

4. $l = m = n$ *and there is an* $\omega \in \mathrm{Aut}(G)$ *such that at least one of*

$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (aba^{-1}, a, c),$$

$$(a', b', c') = \omega \cdot (a, bcb^{-1}, b),$$
$$(a', b', c') = \omega \cdot (c, b, cac^{-1}),$$

$$(a', b', c') = \omega \cdot (b, c, a),$$
$$(a', b', c') = \omega \cdot (c, a, b)$$

holds.

The following theorems about symmetry groups will be used in the classification problem in section 3. The result (i), from [Hur], bounds the order of a symmetry group, and (ii) and (iii), which may be found in [Har], restrict the orders of the automorphisms of a symmetry group.

Theorem 4 *Let G be an (l, m, n)-generated symmetry group of a surface of genus σ. Then:*

i) $|G| \leq 84(\sigma - 1)$,

ii) $l, m, n \leq 4\sigma + 2$, *and*

iii) If p is a prime divisor of l, m, or n, then $p = 2\sigma + 1$ or $p \leq \sigma + 1$.

A symmetry group for which $|G| = 84(\sigma - 1)$ is called a *Hurwitz group*. By calculations using the Riemann-Hurwitz equation, it is seen that if G is a Hurwitz group, then $(l, m, n) = (2, 3, 7)$. It is known that a Hurwitz group exists for $\sigma = 3$ and 7, but not for $\sigma = 2, 4, 5$, or 6 [M].

3 Symmetry and Tiling Groups for Genus 4 and 5

In [Br], Broughton classifies all possible orientation-preserving group actions on surfaces of genus 2 and 3. The difference here is that we limit ourselves to group actions that have three branch points (triangle symmetry groups, as described in the previous section), while there are no such limits in [Br]. However, we are also concerned here with whether or not the symmetry group is also a tiling group, by looking for the appropriate involutary automorphism θ.

To begin the classification for a specific genus, we find all possible values for $|G|$ and (l, m, n) by using the Riemann-Hurwitz equation, which is easily done with a procedure on software such as Maple. So starting with this list of data, we eliminate those cases for which no group exists and find all groups that satisfy the feasible data. The easiest data to start with are those that have one of l, m, or n

equal to $|G|$. For these cases, since we know at least one element of G has order equal to $|G|$, the only possibility is for G to be cyclic. Examples of this can be seen for some of the smaller order groups in the tables in section 5. The next step is to eliminate data by using some of the theorems stated in section 2. The following examples demonstrate these ideas.

Example 5 *For $\sigma = 4$, possible branching data (all will be given in the form $|G|, (l, m, n)$) is $21, (3, 3, 21)$. Without checking the cyclic group of order 21, we may use Theorem 2 to eliminate this as a possibility since $21 > 4\sigma + 2 = 18$.*

Example 6 *We may eliminate the data $\sigma = 4,\ 252, (2, 3, 7)$ and $\sigma = 5,\ 336, (2, 3, 7)$ since there are no Hurwitz groups for genus 4 and 5, by the comments after Theorem 4. Also iii) of Theorem 4 can be used, as in the following example.*

Example 7 *The data $\sigma = 4,\ 56, (2, 4, 7)$ may be eliminated by part iii) Theorem 4, since 7 is prime, and $7 \neq 2\sigma + 1$, and $7 > \sigma + 1$. Similarly, we may use Theorem 4 to eliminate $\sigma = 5,\ 28, (2, 7, 14)$ as a possibility for branching data of a symmetry group.*

The following examples demonstrate how to take advantage of specific values of l, m, or n to classify possible groups satisfying branching data or eliminate the data. It is especially helpful when one of l, m, or n is equal to $|G|/2$ as seen in the next example.

Example 8 *Consider the data $\sigma = 4,\ 32, (2, 4, 16)$. Suppose G is a group corresponding to this data. Then we know that G must have a cyclic subgroup of order 16 since $n = 16$. This subgroup must be normal in G since its index is 2. Since $l = 2$, and any two elements in the generating triple must generate all of G, then there must be an element of order 2 not contained in the cyclic subgroup of order 16 (otherwise only 16 elements are generated by the first and third elements in the generating triple). Therefore we must have $G \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{16}$. We may construct all such groups by finding all elements in $Aut(\mathbb{Z}_{16}) \cong \mathbb{Z}_{16}^*$ of order 2. Then we check each of these groups with Magma, and find that $G \cong \langle x, y \mid x^2 = y^{16} = 1, y^x = y^7\rangle$.*

Example 9 *We will now apply Sylow's theorems to eliminate the data $\sigma = 5,\ 30, (3, 5, 5)$ as possible branching data of a symmetry group. By Sylow's theorems, we know that a group of order 30 must have 1 or 10 Sylow-3 subgroups, and 1 or 6 Sylow-5 subgroups. If we assume that neither of these is 1, then we get that our group has more than 30 elements. Therefore, one of these Sylow subgroups must be normal. However, this would mean the first two elements in the generating triple would generate 15 elements instead of all 30. Therefore, there is no symmetry group corresponding to this data.*

Once the author exhausted the above methods of classification, large group search calculations in Magma were implemented. All non-abelian groups of order less than or equal to 100 (with the exception of 64) are stored in a Magma database as polycyclic-represented groups. Code was written to take a specific $|G|$ and (l, m, n), and search through all of the stored groups of this order to check if they are (l, m, n)-generated. To check for possible abelian groups, it is useful to remember that all of the groups must be generated by two elements, and this fact eliminates many abelian groups of a specific order as a possible symmetry group. Possible abelian groups can be further reduced by considering the cyclic groups contained in G implied by the values of l, m, and n.

After determining that a group G is a symmetry group, we are interested in whether or not G is also a tiling group by looking for the proper involutary automorphism. To do this on Magma, we imbed G in a permutation group, by taking the regular representation of G, call it $\pi(G)$. We then take the normalizer of $\pi(G)$ in the whole symmetric group S_n, where $n = |G|$. It turns out that this gives us the holomorph of G, $\mathrm{Hol}(G)$. That is,

$$N_{S_n}(\pi(G)) \cong \mathrm{Hol}(G) = \mathrm{Aut}(G) \ltimes G$$

(see [DF] or [R]). This calculation makes the automorphisms of G accessible for calculation as permutations. The only problem with this method of obtaining the automorphisms of G is that when$|G|$ is large, then S_n is large, and calculations take a lot of time. The results from the methods used in this section are listed in the tables in section 6.

4 Some Classes of Symmetry Groups

When looking at the tables in section 6 of symmetry and tiling groups for surfaces of genus 4 and 5, we immediately notice a few outstanding patterns. Firstly, there are no non-abelian odd order groups. In fact, there is only one non-abelian symmetry group that has odd order for $2 \leq \sigma \leq 5$, the non-abelian group of order 21, and even that is not a tiling group. The following result reveals that there are in fact infinitely many odd order non-abelian tiling groups.

Theorem 10 *Let $p > 2$ be a prime, and let G be the non-abelian group of order p^3 such that all elements of G have order p. Then G is a tiling group of a surface of genus $\sigma = \frac{1}{2}(p^3 - 3p^2 + 2)$*

Proof. The group G has the presentation $\langle x, y \mid x^p = y^p = 1, [x, y]^x = [x, y]^y = [x, y]\rangle$. It is also known that the group G is *extra-special*, that is, the center of G ($\zeta(G)$), and the derived group of G, ($G' = [G, G]$), coincide (see [R]). Since all elements of G have order p, then we must have $(l, m, n) = (p, p, p)$. Consider the generating triple $(x, y, (xy)^{-1})$. We need an automorphism θ of G such that $\theta(x) = x^{-1}$, $\theta(y) = y^{-1}$, and $\theta^2 = id$. In order for this to be an automorphism, the relations in the presentation still must be valid in the image of θ. First note that

$$[x, y]^x = x^{-1}[x, y]x = [x, y] \iff$$
$$[x, y]^{x^{-1}} = x[x, y]x^{-1} = [x, y].$$

Therefore in G we have $[x, y]^{x^{-1}} = [x, y]^{y^{-1}} = [x, y]$. Now in order for θ to be an automorphism, we must have $[x, y] = [x^{-1}, y^{-1}]$. Now,

$$[x, y] = [x^{-1}, y^{-1}] \iff x^{-1}y^{-1}xy = xyx^{-1}y^{-1} \iff x^{-1}y^{-1}xyyx = xy.$$

Notice that $x^{-1}y^{-1}xy \in G' = \zeta(G)$, so that this element commutes with xy. Then $(x^{-1}y^{-1}xy)(yx) = (yx)(x^{-1}y^{-1}xy) = xy$, and $[x^{-1}, y^{-1}]^{x^{-1}} = [x^{-1}, y^{-1}]^{y^{-1}} = [x^{-1}, y^{-1}]$. It is trivial that $x^{-p} = y^{-p} = 1$, and that $\theta^2 = id$, so θ is the desired automorphism. Therefore G is a tiling group, and the genus of the surface it tiles can be calculated using the Riemann-Hurwitz equation. ■

Looking at the tables in section 6 shows that there is only one other non-tiling symmetry group in the tables besides the aforementioned group of order 21, for $2 \leq \sigma \leq 5$. Thus we might ask is group if we can construct other non-tiling symmetry groups. The following theorem characterizes an entire class of non-tiling symmetry groups of which the group of order 21 is the first example.

Theorem 11 *Let G be a non-abelian group such that $|G| = pq$, where p and q are distinct odd primes. Then G is a symmetry group but is not a tiling group.*

Proof. By using Sylow's Theorem, we see that $G \cong \mathbb{Z}_p \ltimes \mathbb{Z}_q$, where $q \equiv 1 (\text{mod } p)$, (possibly reversing the roles of p and q). By finding an element k_0 of $\text{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^*$, we obtain a presentation for G:

$$G \cong \langle x, y \mid x^p = y^q = 1, x^{-1}yx = y^{k_0} \rangle$$

where $k_0^p \equiv 1 (\text{mod } p)$ and $k_0 \not\equiv 1 \ (\text{mod } q)$. In fact, it can be easily shown that for any two elements $z, w \in G$ of orders p and q respectively that $z^{-1}wz = w^k$ where k is one of the $p - 1$ non-trivial solutions to $k^p \equiv 1 (\text{mod } q)$. Now, if G is to be (l, m, n)-generated, then we must use only p or q for l, m, and n, since if one is pq, we get G is cyclic. We also know that G should be generated by two of the elements in our (a, b, c) generating triple. But G has a unique normal subgroup of order q by Sylow's theorem, so if two of l, m, or n is q, then not all of G is generated by the two elements of order q. Therefore we must have $(l, m, n) = (p, p, q)$ or (p, p, p).

First let us deal with the case (p, p, q) and let (a, b, c) be a (p, p, q)-generating triple. As noted above $a^{-1}ca = c^k$ for k as above. Therefore, we have G is a symmetry group of a surface of genus $\sigma = \frac{1}{2}(pq - 2q - p + 2)$, by the Riemann-Hurwitz equation. A possible generating triple is $(x, (yx)^{-1}, y)$. In [Br], it is proved that this is the only generating triple up to automorphism. Now assume that G is a tiling group, and that we have an automorphism ψ of G such that $\psi(a) = a^{-1}$, $\psi(b) = b^{-1}$, and $\psi^2 = id$. Remembering that $abc = 1$ it follows that $1 = \psi(1) = \psi(abc) = \psi(a)\psi(b)\psi(c) = a^{-1}b^{-1}\psi(c)$. Thus

$$\psi(c) = ba = a^{-1}c^{-1}a = c^{-k}.$$

Now we must have $\psi^2(c) = c$, while

$$\psi(\psi(c)) = \psi(c^{-k}) = (\psi(c))^{-kj} = (y^{-k})^{-k} = c^{k^2}. \tag{3}$$

But if $c^{k^2} = c$, then since $c^q = 1$, we have $k^2 \equiv 1(\text{mod } q)$.However k is a non-trivial solution of $k^p \equiv 1 \,(\text{mod}\, q)$. Since 2 and p are relatively prime, $k \equiv 1 \,(\text{mod}\, q)$, a contradiction, and so such an automorphism ψ cannot exist. Therefore, G is not a tiling group.

Now consider the case $(l, m, n) = (p, p, p)$ Many examples of (p, p, p)-generating triples (a, b, c) exist, say $(x, xy, y^{-1}x^{p-2})$ for this symmetry group. The genus is $\frac{1}{2}(pq - 3q - 2)$. By considering the map $\mathbb{Z}_q \longrightarrow G \longrightarrow \mathbb{Z}_p$ and the images $\overline{a}, \overline{b}, \overline{c}$ of a, b, c in $\mathbb{Z}_p$ we see $\overline{b} = \overline{a}^s$ for some s. It follows that $b = da^s$ for some $d \in \mathbb{Z}_q$. By reusing a previous argument we get an equation similar to 3:

$$\psi(d) = \psi(ba^{-s}) = b^{-1}a^s = b^{-1}d^{-1}b = d^{-k},$$

for a k as above, and we get another contradiction, finishing our proof. ■

The reader should note that the above theorem may be adjusted to apply to some non-abelian groups of the form $\mathbb{Z}_4 \ltimes \mathbb{Z}_p$ with the specific triple $(4, 4, p)$, where $p > 3$ is prime, including the group of order 20 that is in Table 3 in section 6. This result is not as interesting since it is for a specific triple for a class of groups, and not for all possible triples (see section 5, question 3).

5 Further Questions

There is a lot known about p-groups, and it would be interesting to know more about their role as tiling groups. This leads to the first question, which would be a generalization of Theorem 10.

Question 1. Let $p > 2$ be a prime. Then for all $n > 2$, does there exists a non-abelian tiling group G such that $|G| = p^n$?

Proposition 2, along with not being able to find a counterexample, prompts the second question.

Question 2. If G is non-abelian and$|G|$ is odd and square-free, then can G be a tiling group?

The nature and abundance of non-tiling symmetry groups is not completely understood. If the answer to the next question is negative, then it would be a powerful tool in finding more classes of non-tiling symmetry groups.

Question 3. If G is a non-tiling symmetry group for the triple (l, m, n), then can there exist another triple (l', m', n') for which G is a tiling group?

6 Tables of Symmetry and Tiling Groups

This section contains tables of symmetry and tiling groups found for surfaces of genus 4 and 5. The tables give the order of the group, the (l, m, n) triple for the group, the group or a presentation of the group, and whether or not the symmetry group is also a tiling group. The presentations that have more than two generators are the polycyclic presentations of the group, and the conjugation relations that are left out are those that say when two generators commute.

Notes on table 3. It is unknown whether there are more groups of order 120 that are (2, 4, 5) generated, and the classification for the last two entries is incomplete. However, there can be no additional symmetry groups for genus 4 surfaces.

Notes on table 4. This classification is incomplete for the last four entries in the table above, groups for each of these may or may not exist. It is also unknown if there are other groups of order 120 that are (2, 3, 10) generated. However, it is certain that there can be no other genus 5 symmetry groups.

Table 1. Symmetry and Tiling Groups for Genus 2 Surfaces

$\|G\|$	(l, m, n)	G	Tileable?
5	(5, 5, 5)	$\mathbb{Z}_5$	Yes
6	(3, 6, 6)	$\mathbb{Z}_6$	Yes
8	(2, 8, 8)	$\mathbb{Z}_8$	Yes
8	(4, 4, 4)	$\langle x, y \| x^4 = y^4 = 1, x^2 = y^2, y^x = y^{-1} \rangle$	Yes
10	(2, 5, 10)	$\mathbb{Z}_{10}$	Yes
12	(2, 6, 6)	$\mathbb{Z}_2 \times \mathbb{Z}_6$	Yes
12	(3, 4, 4)	$\langle x, y \| x^4 = y^3 = 1, y^x = y^{-1} \rangle$	Yes
16	(2, 4, 8)	$\langle x, y \| x^2 = y^8 = 1, y^x = y^3 \rangle$	Yes
24	(2, 4, 6)	$\langle x, y, z, w \mid x^2 = y^2 = z^2 = w^3 = 1,$	Yes
		$z^x = zy, w^x = w^{-1} \rangle$	
24	(3, 3, 4)	$SL_2(3)$	Yes
48	(2, 3, 8)	$GL_2(3)$	Yes

Table 2. Symmetry and Tiling Groups for Genus 3 Surfaces

$\lvert G\rvert$	(l, m, n)	G	Tileable?
7	(7, 7, 7)	$\mathbb{Z}_7$	Yes
8	(4, 8, 8)	$\mathbb{Z}_8$	Yes
9	(3, 9, 9)	$\mathbb{Z}_9$	Yes
12	(2, 12, 12)	$\mathbb{Z}_{12}$	Yes
12	(3, 4, 12)	$\mathbb{Z}_{12}$	Yes
12	(4, 6, 6)	$\langle x, y \mid x^4 = y^3 = 1, y^x = y^{-1}\rangle$	Yes
14	(2, 7, 14)	$\mathbb{Z}_{14}$	Yes
16	(2, 8, 8)	$\mathbb{Z}_2 \times \mathbb{Z}_8$	Yes
16	(2, 8, 8)	$\langle x, y \mid x^2 = y^8 = 1, y^x = y^5\rangle$	Yes
16	(4, 4, 4)	$\mathbb{Z}_4 \times \mathbb{Z}_4$	Yes
16	(4, 4, 4)	$\langle x, y \mid x^4 = y^4 = 1, y^x = y^{-1}\rangle$	Yes
21	(3, 3, 7)	$\langle x, y \mid x^3 = y^7 = 1, y^x = y^2\rangle$	**NO**
24	(2, 4, 12)	$\langle x, y \mid x^4 = y^{12} = 1, y^x = y^5\rangle$	Yes
24	(2, 6, 6)	$\mathbb{Z}_2 \times A_4$	Yes
24	(3, 3, 6)	$SL_2(3)$	Yes
24	(3, 4, 4)	S_4	Yes
32	(2, 4, 8)	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1, z^x = yz^2\rangle$	Yes
32	(2, 4, 8)	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1, z^y = z^5,$ $y^x = yz^4, z^x = yz^3\rangle$	Yes
48	(2, 4, 6)	$\mathbb{Z}_2 \times S_4$	Yes
48	(3, 3, 4)	$\langle x, y, z \mid x^3 = y^4 = z^4 = 1, y^x = z, z^x = (yz)^{-1}\rangle$	Yes
96	(2, 3, 8)	$\langle x, y, z, w \mid x^2 = y^3 = z^4 = w^4 = 1, y^x = y^2,$ $z^x = z^{-1}w, z^y = z^2w, w^y = z\rangle$	Yes
168	(2, 3, 7)	$PSL_2(7)$	Yes

Table 3. Symmetry and Tiling Groups for Genus 4 Surfaces

$\|G\|$	(l, m, n)	G	Tileable?
9	(9, 9, 9)	$\mathbb{Z}_9$	Yes
10	(5, 10, 10)	$\mathbb{Z}_{10}$	Yes
12	(4, 6, 12)	$\mathbb{Z}_{12}$	Yes
12	(3, 12, 12)	$\mathbb{Z}_{12}$	Yes
15	(3, 5, 15)	$\mathbb{Z}_{15}$	Yes
16	(2, 16, 16)	$\mathbb{Z}_{16}$	Yes
16	(4, 4, 8)	$\langle x, y, z \mid x^2 = z^2, y^2 = z^3, y^x = yz, z^x = z^3\rangle$	Yes
18	(2, 9, 18)	$\mathbb{Z}_{18}$	Yes
18	(3, 6, 6)	$\mathbb{Z}_6 \times \mathbb{Z}_3$	Yes
18	(3, 6, 6)	$S_3 \times \mathbb{Z}_3$	Yes
20	(2, 10, 10)	$\mathbb{Z}_{10} \times \mathbb{Z}_2$	Yes
20	(4, 4, 5)	$\langle x, y \mid x^4 = y^5 = 1, y^x = y^{-1}\rangle$	Yes
20	(4, 4, 5)	$\langle x, y \mid x^4 = y^5 = 1, y^x = y^2\rangle$	**NO**
24	(2, 6, 12)	$\langle x, y \mid x^2 = y^{12} = 1, y^x = y^7\rangle$	Yes
24	(3, 4, 6)	$\langle x, y, z \mid x^3 = y^4 = z^4 = 1, y^x = z, z^x = yz^3, z^y = z^3,$ $[y^2, x] = 1\rangle$	Yes
32	(2, 4, 16)	$\langle x, y \mid x^2 = y^{16} = 1, y^x = y^7\rangle$	Yes
36	(3, 4, 4)	$\langle x, y, z, w \mid x^2 = y, y^2 = z^3 = w^3 = 1, z^x = w^2, z^y = z^2,$ $w^x = z, w^y = w^2\rangle$	Yes
36	(3, 3, 6)	$\langle x, y, z, w \mid x^3 = y^3 = z^2 = w^2 = 1, z^x = zw, w^x = z\rangle$	Yes
36	(2, 6, 6)	$\langle x, y, z, w \mid x^2 = y^2 = z^3 = w^3 = 1, w^x = w^2\rangle$	Yes
36	(2, 6, 6)	$\langle x, y, z, w \mid x^2 = y^2 = z^3 = w^3 = 1, z^x = z^2, w^y = w^2\rangle$	Yes
40	(2, 4, 10)	$\langle x, y, z, w \mid x^2 = 1, y^2 = z, z^2 = w^5 = 1, y^x = yz, w^y = w^4\rangle$	Yes
60	(2, 5, 5)	A_5	Yes
72	(2, 4, 6)	$\langle x, y, z, w, v \mid x^2 = z, y^2 = z^3 = w^3 = v^3 = 1, y^x = yz,$ $w^x = wv, w^y = w^2, w^z = w^2, v^x = w^2v^2, v^y = wv, v^z = v^2\rangle$	Yes
72	(2, 3, 12)	$\langle x, y, z, w, v \mid x^2 = y^3 = z^3 = w^2 = v^2 = 1, z^x = z^3,$ $w^z = wv, v^x = wv, v^z = w\rangle$	Yes
120	(2, 4, 5)	S_5	Yes
108	(2, 3, 9)		
144	(2, 3, 8)		

Table 4. Symmetry and Tiling Groups for Genus 5 Surfaces

$\lvert G\rvert$	(l, m, n)	G	Tileable?
11	(11, 11, 11)	$\mathbb{Z}_{11}$	Yes
12	(6, 12, 12)	$\mathbb{Z}_{12}$	Yes
15	(3, 15, 15)	$\mathbb{Z}_{15}$	Yes
16	(4, 8, 8)	$\mathbb{Z}_8 \times \mathbb{Z}_2$	Yes
16	(4, 8, 8)	$\langle x, y, z, w \mid x^2 = z, y^2 = w^2 = 1, z^2 = w, y^x = yw\rangle$	Yes
20	(2, 20, 20)	$\mathbb{Z}_{20}$	Yes
20	(4, 4, 10)	$\langle x, y \mid x^4 = y^5 = 1, y^x = y^{-1}\rangle$	Yes
22	(2, 11, 22)	$\mathbb{Z}_{22}$	Yes
24	(2, 12, 12)	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	Yes
24	(4, 4, 6)	$\mathbb{Z}_2 \times (\mathbb{Z}_4 \ltimes \mathbb{Z}_3)$	Yes
30	(2, 6, 15)	$\mathbb{Z}_3 \times D_5$	Yes
32	(4, 4, 4)	$\langle x, y, z, w, v \mid x^2 = w, y^2 = z^2 = w^2 = v^2 = 1, y^x = yz,$ $z^x = zv, w^y = wv\rangle$	Yes
32	(4, 4, 4)	$\langle x, y, z, w, v \mid x^2 = w, y^2 = v, z^2 = w^2 = v^2 = 1, y^x = yz\rangle$	Yes
32	(2, 8, 8)	$\langle x, y, z, w, v \mid x^2 = w, y^2 = z^2 = v^2 = 1, w^2 = v, y^x = yz\rangle$	Yes
32	(2, 8, 8)	$\langle x, y, z, w, v \mid x^2 = w, y^2 = z^2 = v^2 = 1, w^2 = v, y^x = yz,$ $z^x = zv, w^y = wv\rangle$	Yes
40	(2, 4, 20)	$\langle x, y \mid x^2 = y^{20} = 1, y^x = y^9\rangle$	Yes
48	(3, 4, 4)	$\langle x, y, z, w, v \mid x^2 = y, y^2 = z^3 = w^2 = v^2 = 1, z^x = z^3,$ $w^z = wv, v^x = wv, v^2 = 2\rangle$	Yes
48	(2, 6, 6)	$\langle x, y, z, w, v \mid x^2 = y^2 = z^3 = w^2 = v^2, w^z = wv\rangle$	Yes
48	(2, 4, 12)	$\langle x, y, z, w, v \mid x^2 = z, y^2 = z^2 = w^2 = v^3 = 1, y^x = yw,$ $v^y = v^z\rangle$	Yes
60	(3, 3, 5)	A_5	Yes
80	(2, 5, 5)	$\langle x, y, z, w, v \mid x^5 = y^2 = z^2 = w^2 = v^2 = 1, y^x = w,$ $z^x = yzv, w^x = zw, v^x = wv\rangle$	Yes
96	(3, 3, 4)	$\langle x, y, z, w, v, u \mid x^3 = w^2 = v^2 = u^6 = 1, y^2 = w, z^2 = v,$ $y^x = yz, z^x = yvu, z^y = zu, w^x = wvu, z^x = w\rangle$	Yes
96	(2, 4, 6)	$\langle x, y, z, w, v, u \mid x^2 = y^2 = z^2 = w^3 = v^2 = u^2 = 1,$ $y^x = yz, w^y = w^2, v^w = vu, u^y = vu, u^w = v\rangle$	Yes
120	(2, 3, 10)	$\mathbb{Z}_2 \times A_5$	Yes
64	(2, 4, 8)		
144	(2, 3, 9)		
160	(2, 4, 5)		
192	(2, 3, 8)		

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