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Abstract

Let S be a compact Riemann surface without boundary. A *symmetry* θ of S is an anti-conformal, involutary automorphism. The fixed point set of θ is a disjoint union of circles, each of which is called an *oval* of θ. A method is presented for counting the ovals of a symmetry when S admits a large group G of automorphisms, normalized by θ. The method involves only calculations in G, based on the geometric description of S/G, and the knowledge of the action of θ on G. As an illustrative example, the family of generic, symmetric surfaces of genus three with conformal automorphism group Σ_4 is constructed as several 1-parameter families.

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Contents

1 Introduction

Let S be a closed Riemann surface of genus $\sigma \geq 2$, admitting a known group of conformal automorphisms G. A *symmetry* θ of S is an anti-conformal, involutary automorphism. The fixed point subset S_θ of θ is a disjoint union of circles, each of which we shall call an oval of θ. If S_θ is non-empty then we call θ a *reflection* and call S_θ the *mirror* of θ. In this paper we shall present a method for counting the ovals of a symmetry when θ normalizes the group G. Our methods involve only calculations in the finite group G, the topology of the quotient surface $T \simeq S/G$, and the action of θ on G. We are particularly interested in those cases where $G = \mathrm{Aut}(S)$, the conformal automorphism group of S, and G is fairly large in comparison to σ, so that T is a surface with simple structure, e.g., a sphere with a small number of branch points.

Here are two motivating scenarios in which symmetries occur. First the surface S may arise as a complex algebraic curve defined as the complex solutions of polynomial equations in several variables. If the coefficients are real then complex conjugation leaves the curve invariant and is a symmetry of the surface. The fixed points of the symmetry are the real points of the curve and the number of ovals is the number of components of the curve. Each different conjugacy class of symmetries yields a different real form of the curve. The second scenario is a tiling of the surface by polygons. Symmetries arise from reflections in the edges of polygons. In this case we may take the group G to be the set of all transformations of the surface which are a product of an even number of reflections, called the conformal tiling group. If G is particularly large then the tiling group will act transitively on the tiles so that the surface may be built from a set of congruent tiles according to rules encoded in the group.

Let $G^* = \langle G, \theta \rangle$. As θ normalizes G, G has index 2 in G^* and θ induces a symmetry ψ on T, yielding a Klein surface $\overline{T} = T/\langle \psi \rangle = S/G^*$. From the boundary $\partial(\overline{T})$ of $\overline{T}$ we shall construct the *dissected boundary* $\partial^d(\overline{T})$ (see §2). An analysis of the components of the dissected boundary will assist us in determining the conjugacy classes of symmetries with ovals, the stabilizers of ovals, and the number of ovals of a symmetry. Formulas for computing the number of ovals in terms of indices of oval stabilizers in the G-centralizer of θ are given in §3. To effectively use the G-action in calculating the number of ovals a description of the G-action in terms of the topology the quotient surface T is required. We do this in §4, using the idea of generating vectors, which are constructed from markings on T. We also use generating vectors to give a combined group-theoretical and topological method for determining all the conjugacy classes of symmetries on S that normalize G, as well as determining the action of θ on G.

The symmetry θ induces a tiling on the hyperbolic plane, the universal cover of S. We introduce and analyze this tiling in §5 and then in §6 apply the results to compute the orders of oval stabilizers, by calculating the orders of certain elements of G. These elements are determined from generating vectors using certain universal words ("magic words") depending only on the marking on T. This completes the missing information from the formulas in §3. We illustrate the discussion in §4, §5 and §6 by explicit examples of symmetric surfaces of genus 3 with $\mathbb{Z}_3$-action, a simple though non-trivial case. Finally, in §7, in a more comprehensive example, we construct several one parameter families of generic, symmetric surfaces of genus 3 with Σ_4 as conformal automorphism group. The conjugacy classes of reflections and the number of ovals of each class of reflections is calculated for all these families. The example is chosen since it yields new results and illustrates most of the complexities that arise without being overwhelming.

The main results of the paper are:

- the oval formulas (Theorems 7 and 8 and Corollary 28),
- a general criterion for the existence of symmetries, and methods for determining conjugacy classes of symmetries (Theorem 12 and Corollary 13),

- the construction of the "magic words" for computing stabilizers of ovals (Theorem 27),
- a brief description of translating symmetric surface tiling problems into group theoretic problems (Remark 18 in §4), and
- the comprehensive example of symmetric surfaces of genus 3 with Σ_4 as conformal automorphism group (see §7 and Tables 7.1, 7.2, 7.3 and 7.4).

These results extend to a more general setting the results of Singerman [6] and others who consider the case when S/G is a sphere with three branch points. These results are conveniently and extensively presented in [5], especially the oval formula and "magic words" for oval stabilizers. Our work was also strongly motivated by [4] in which analogous methods were used count ovals on surfaces with $PSL_2(q)$ action.

In addition to extending the results of previous work we have adopted a somewhat different though equivalent approach. In previous work most authors worked with the covering group on the universal covering space and the group's presentation. Here we work directly the finite group and strongly use the topology of S/G. (We do, however use Fuchsian groups and the universal cover for proofs!) We regard the group G and the topology of S/G as the primary data of the surface and formulate the main results in this context. The reason for this approach is that the strength of the influence of the structure of G on the geometry of S is inversely related to the simplicity of the topology and geometry of S/G and S/G^*. The approach also lends itself very well to computer calculation of fairly complex examples when the topology of S/G is simple.

Throughout the paper will freely use standard facts about Klein surfaces, Fuchsian groups, and NEC groups. A suitable background reference for this is [1].

Notation and Terms We adopt the following notation throughout the paper. For an arbitrary conformal G-action on a surface S, let $\Gamma \triangleleft \Lambda \subset \mathrm{Aut}(\mathbb{H})$ be uniformizing pair of Fuchsian groups for S and the G-action. Thus we will have an epimorphism $\eta : \Lambda \to G$ with kernel Γ, such that Γ is torsion free, and $S \simeq \mathbb{H}/\Gamma$. Furthermore G acts S via the inverse of the induced isomorphism $\overline{\eta} : \Lambda/\Gamma \to G$, i.e., the map $\epsilon = \overline{\eta}^{-1} : G \to \Lambda/\Gamma \subset \mathrm{Aut}(S)$ is a monomorphism and the action is defined by $gx = \epsilon(g)(x)$ for $g \in G$ and $x \in S$. If ω is an automorphism of G we may twist the action via the epimorphism $\omega \circ \eta : \Lambda \to G$. However, the same kernel, surface and subgroup of $\mathrm{Aut}(S)$ is determined; so they are considered equivalent for our purposes.

Let θ_0 be a fixed symmetry of S normalizing G, and let ψ be the map induced on T by θ_0. Define $G^* = \langle \theta_0, G \rangle = \langle \theta_0 \rangle \ltimes G$, and let θ be any other symmetry in G^*. Let Λ^* be the unique NEC group containing Λ and corresponding to G^*, the map η may be extended $\eta : \Lambda^* \to G^*$. The map ψ is a symmetry of T, we define the quotient $\overline{T} = T/\langle \psi \rangle = S/G^* = \mathbb{H}/\Lambda^*$. The surface $\overline{T}$ is a Klein surface and has a boundary $\partial(\overline{T})$ if and only if some symmetry in G^* has ovals.

A point $\bar{x} \in \overline{T}$ is called *periodic* or a *branch point*, if there is a point $x \in \mathbb{H}$ projecting to $\bar{x}$ such that $\Lambda_x = \{\gamma \in \Lambda : \gamma x = x\}$ is non-trivial. This stabilizer is cyclic and the order of Λ_x is called the *period* of $\bar{x}$. Obviously, the periodic points are the images of the branch points on T and the periods are the same. We call the point $Q \in T$ *symmetric* or *involutary* according to whether $\psi(Q) = Q$ or $\psi(Q) \neq Q$. In the latter case $\{Q, \psi(Q)\}$ is called an involutary pair.

Throughout the paper $\mathbb{Z}_n$, D_n and Σ_n denote the cyclic group of order n, the dihedral group of order $2n$ and the symmetric group on n symbols, respectively. Finally, for $g, x \in G$, $\mathrm{Ad}_g(x) = gxg^{-1}$.

2 The dissected boundary

From the boundary $\partial(\overline{T})$ of $\overline{T}$ we construct the *dissected boundary* $\partial^d(\overline{T})$ consisting of a collection of boundary components and subintervals of boundary components of $\partial(\overline{T})$. Include in $\partial^d(\overline{T})$ each boundary component of $\partial(\overline{T})$ without even periodic points, including those without any periodic points at all. If a boundary component of $\overline{T}$ has even periods then cut this circle at all even periodic points, complete each open interval so obtained to a closed interval, and also include these intervals in $\partial^d(\overline{T})$. We write $\partial^d(\overline{T}) = \{C_1, \ldots, C_s, I_1, \ldots, I_t\}$ $= \{B_0, \ldots, B_{r-1}\}$,where C_i is a circle and I_j is an interval. Mark each object in $\partial^d(\overline{T})$ with its periods derived from $\overline{T}$. Note that the elements of $\partial^d(\overline{T})$ are disjoint except possibly that some intervals meet at endpoints. Also note that for each $B \in \partial^d(\overline{T})$ the natural map $B \to \overline{T}$ induces a homeomorphism of B onto its image except where there is exactly one even periodic point on a boundary component of $\overline{T}$. Let $\bar{B}$ denote the image of $B \in \partial^d(\overline{T})$ in $\overline{T}$, then

$$\partial(\overline{T}) = \bigcup_{B \in \partial^d(\overline{T})} \bar{B},$$

hence the name dissected boundary.

The most common and group theoretically interesting case is when T is a sphere. In this case ψ is either conjugate to a reflection in a great circle or conjugate to the antipodal map. Reflection in a circle yields a disc, the antipodal map yields the projective plane which has no boundary. If we get the projective plane, none of the symmetries in G^* have ovals – e.g., curves defined over $\mathbb{R}$ all of whose real forms are empty. The symmetry ψ must permute the branch points on T arising from the G-action (see §4), and this limits the possibilities for $\overline{T}$ as the examples following illustrate.

Example 1 Triangular Actions. *A triangular action is one in which the quotient surface S/G is a sphere with three branch points with orders l, m, n. Since there is an odd number of branch points, ψ must fix one of them and so ψ is a reflection in a great circle. This is by far the most common and interesting type of action, and also the simplest to analyze. The quotient $\overline{T}$ must be orientable and all the branch points must lie on the boundary or two of the branch points form an involutary pair of equal order.. In the first situation there are four possible cases for the dissected boundary: a circle with three odd order points; a segment with two interior odd order points and endpoints of the same even order; two segments, one with a single interior odd point; three segments with no interior points. In the involutary pair case the boundary must be a circle with one odd order branch or a segment with the same even order branch point at the ends and no interior branch points.*

Given three points on the sphere we may always construct the symmetry ψ which moves the points as required. Taking the compactified complex plane as the model of the sphere, we may map the three branch points to $1, -1$ and ∞, and select $z \rightarrow \bar{z}$ or $z \rightarrow -\bar{z}$ as our symmetries.

Example 2 Sphere with 4 branch points. *If ψ is a reflection then four, two or no branch points lie on the fixed point circle of ψ, the other branch points must come in matched pairs. If there are four boundary branch points we have five cases: a circle with all odd orders with up to six different cyclic orderings of the points, a segment with the same even branching order at the endpoints and up to three different orderings of the three interior odd order points, two segments one with two interior odd order points (and two possible orderings branch orders along the segment) the other segment has no interior points; two segments with one interior odd order point each, three segments exactly one of which has an interior odd order point; and four segments. We leave to the reader the description of the remaining cases.*

If ψ is the antipodal map the branch points must come in involutary pairs of equal order. Though we are not considering symmetries without ovals in this paper but they can be easily handled by the methods of this paper.

Not all selections of four points on the sphere admit symmetries. For example in all the cases above the four points must lie on the same circle and this imposes a constraint. Some of the various possibilities are discussed in detail in §7 where we look at Σ_4 actions on genus 3 surfaces and T has four branch points with orders 2, 2, 2, 3.

The following propositions clarifies the relationship between reflections in Λ^* and components of the dissected boundary.

Proposition 3 *Let Λ^* and $\partial^d(\overline{T})$ be as defined above. Then, the conjugacy classes of reflections in Λ^* are in 1-1 correspondence to the elements of $\partial^d(\overline{T})$.*

To prove this proposition we will need the following basic proposition, which is easily proven by covering space arguments.

Proposition 4 *Let $\mathcal{O}$ be an oval of a symmetry $\theta \in G^*$. Then there is a reflection $\tilde{\theta} \in \Lambda^*$ covering θ such that the fixed line $L_{\tilde{\theta}}$ of $\tilde{\theta}$ projects to the oval $\mathcal{O}$ under the covering map $\mathbb{H} \to \mathbb{H}/\Gamma = S$.*

Proof of Proposition 3 For each reflection ϕ in Λ^* we will show below that the image of the fixed line L_ϕ, under the map $\mathbb{H} \to \mathbb{H}/\Lambda^* = \overline{T}$, is $\bar{B}$ for a uniquely determined B in $\partial^d(\overline{T})$. Now suppose that ϕ' is any other reflection in Λ^*. Since

$$\gamma L_\phi = L_{\gamma\phi\gamma^{-1}}, \text{ for } \gamma \in \mathrm{Aut}^*(\mathbb{H}),$$

then ϕ and ϕ' are Λ^*-conjugate if and only if the fixed lines L_ϕ, $L_{\phi'}$ are Λ^*-equivalent. In turn, the fixed lines project to the same image in $\partial(\overline{T})$ if and only if they are Λ^*-equivalent. It follows that each conjugacy class of reflections in Λ^* determines a unique element of $\partial^d(\overline{T})$ and that distinct classes are associated to distinct elements in $\partial^d(\overline{T})$. Since each non-periodic point of $\partial(\overline{T})$ arises from a non-periodic point of some fixed line L_ϕ, the correspondence is bijective.

Now let us show that L_ϕ maps onto a unique $\bar{B}$. Obviously, L_ϕ maps into $\partial(\overline{T})$. Pick a point on L_ϕ which is not a periodic and proceed along L_ϕ until the first periodic point (if any), say x, of period n, is reached. Corresponding to the segment traced out on L_ϕ, a path will be traced out along a component of $\partial(\overline{T})$ reaching the image point $\bar{x}$. If there are no periodic points on L_ϕ the path traced out simply winds around the boundary circle infinitely many times.

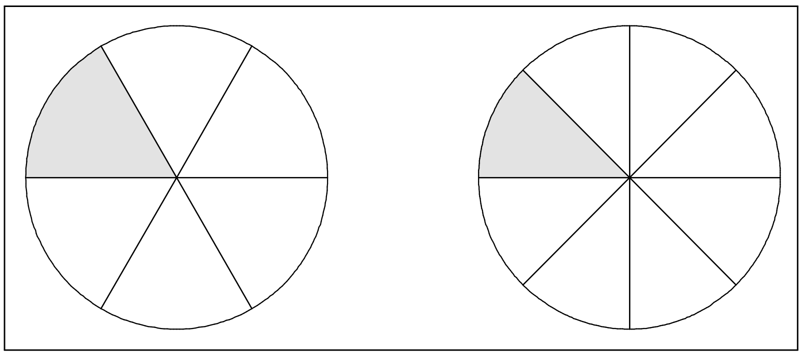

Figure 1. Local quotients of order 3 and 4

If there are periodic points on L_ϕ the behavior of the image path as we pass through the periodic point depends only on the parity of the period. The behavior of the image is illustrated by the cases of periodic points of order 3 and 4 in Figure 1. The point x has a small neighbourhood such that local action of Λ^* is topologically equivalent to a dihedral group D_n acting on circle. In Figure 1, the circle has been subdivided into $2n$ sectors which D_n permutes simply transitively. The distinguished sector is a fundamental region, and the union of the two radial edges of this sector maps homeomorphically onto a small neighbourhood of $\bar{x}$ in $\partial(\overline{T})$. Now in the figure suppose that L_ϕ is the x-axis. After we pass through x at the centre we move along the segment which is on the positive part of the x-axis. This segment is D_n-equivalent to exactly one of the two radial edges of the distinguished sector, and the direction of travel is from the origin. If n is even the equivalent edge is the one along the negative x-axis otherwise it is the other edge. Using the distinguished sector as the local model of $\overline{T}$ at $\bar{x}$, we see that if n is odd the path will travel in along the negative x-axis and the out along the other edge. The image path in $\partial(\overline{T})$ "passes through" $\bar{x}$. On the other hand if n is even then the path comes in along the negative x-axis and leaves along the same edge, so the image path "bounces back" from $\bar{x}$. Thus, if there are no even points then the image path winds around the circle infinitely often giving us one of the circles in the dissected boundary. If their are even points then image path bounces back and forth along one of the segments we created in the dissected boundary. In either case the image of L_ϕ is a unique $\bar{B}$ and all is now proven. ■

Fusion of symmetries Let θ be any symmetry in G^* with at least one oval $\mathcal{O}$. As in the proof of Proposition 3, there is a unique $B \in \partial^d(\overline{T})$ such that $\mathcal{O}$ maps onto $\overline{B} \subseteq \overline{T}$ under the projection $S \to \overline{T}$. For each $B \in \partial^d(\overline{T})$, define $\mathcal{O}(\theta, B)$ to be the set of ovals of θ that map to $\overline{B}$ under $S \to \overline{T}$, and say that these ovals have type B. It is possible for reflection to have more than one type of oval. To this end let $B_0, \ldots, B_{r-1}$ be the various components of the dissected boundary, let $\widetilde{\theta}_0, \ldots, \widetilde{\theta}_{r-1}$ be reflections in Λ^* that correspond to $B_0, \ldots, B_{r-1}$ via Proposition 3. Let $\theta_0, \ldots, \theta_{r-1}$ be the images of $\widetilde{\theta}_0, \ldots, \widetilde{\theta}_{r-1}$ under the map $\Lambda^* \to G^*$, so that B_0 is associated to our distinguished reflection θ_0 via its covering reflection $\widetilde{\theta}_0$. Now in Λ^* the reflections $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_r$ are pairwise non-conjugate, however their conjugacy classes may fuse under the map $\Lambda^* \to G^*$, i.e., some of the images $\theta_1, \ldots, \theta_r$ may be conjugate in G^*. The exact relationship between this "fusion of symmetries" and multiple oval types of a symmetry is contained in the following proposition.

Proposition 5 *Let $\theta \in G^*$ be a symmetry and let $B_0, \ldots, B_{r-1}$ and $\theta_0, \ldots, \theta_{r-1}$ and other notation be as above. Then, the following hold.*

i) If θ has a fixed point subset then at least one $\mathcal{O}(\theta, B_i)$ is non-empty.

ii) The set $\mathcal{O}(\theta, B_i)$ is non-empty if and only θ is conjugate to θ_i.

iii) The element $\theta \in G^*$ is a symmetry if and only if $\theta = \theta_0 d$ with $\theta_0 d \theta_0 = d^{-1}$.

Theorem 6 *iv) Two symmetries $\theta = \theta_0 d$ and $\theta' = \theta_0 d'$ are conjugate if and only if there is a $g \in G$ with $d' = \theta_0 g \theta_0 d g^{-1}$.*

Proof. The statements *i*) and *ii*) are simple exercises, using Propositions 3 and 4. The statements *iii*) and *iv*) follow from the fact that all orientation reversing elements of G^* form the coset $\theta_0 G$ and that a symmetry $\theta_0 d$ must satisfy $1 = (\theta_0 d)^2 = (\theta_0 d \theta_0) d$. ■

3 The oval formulas

The oval formulas we present are equivalent the ones presented in [5], though they apply in a more general setting and are given explicitly in terms of the G-action and the dissected boundary. Let θ be any symmetry in G^* with at least one oval $\mathcal{O}$. The following theorem gives us a way of counting ovals.

Theorem 7 *Let $\theta \in G^*$ be a symmetry, let $\mathcal{O}(\theta, B)$, $\partial^d(\overline{T}) = \{B_0, \ldots, B_{r-1}\}$, and other notation be as above. Then, the following hold.*

i) The number k of ovals of θ is given by

$$k = \sum_{i=0}^{r-1} |\mathcal{O}(\theta, B_i)|.$$

ii) If $\mathcal{O}(\theta, B)$ is non-empty then the size of $\mathcal{O}(\theta, B)$ is given by:

$$|\mathcal{O}(\theta, B)| = \frac{|\mathrm{Cent}_G(\theta)|}{|\mathrm{Stab}_G(\mathcal{O})|}.$$

where $\mathcal{O}$ is an oval of θ projecting to B.

Proof. Statement *i*) follows from Proposition 5. To prove *ii*) first we show that the subgroup of G^* permuting the ovals of θ is $\mathrm{Cent}_{G^*}(\theta)$. Clearly the centralizer of θ permutes the ovals of θ. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two ovals of θ and $g \in G^*$ satisfies $g\mathcal{O}_1 = \mathcal{O}_2$, then both $g\theta g^{-1}$ and θ fix $\mathcal{O}_2$. The map $g\theta g^{-1}\theta^{-1}$ is then a conformal map fixing a circle, so it must be the identity by analytic continuation, and thus g centralizes θ. Next, from Proposition 4 it follows that any two ovals in $\mathcal{O}(\theta, B)$ are G^*-equivalent since they are covered by Λ^*-equivalent reflecting lines in $\mathbb{H}$. It follows then that $\mathrm{Cent_{G^*}}(\theta)$ transitively permutes the ovals in $\mathcal{O}(\theta, B)$. Since $\mathrm{Cent_{G^*}}(\theta) = \langle\theta\rangle \times \mathrm{Cent_G}(\theta)$, then it follows that $\mathrm{Cent_G}(\theta)$ also transitively permutes the ovals in $\mathcal{O}(\theta, B)$. The statement now follows from the orbit-stabilizer theorem. ■

Theorem 8 *Let $\mathcal{O}$ and B be as in Theorem 7. Then:*

i) $\mathrm{Stab}_G(\mathcal{O})$ *acts effectively on* $\mathcal{O}$*,*

ii) $\mathcal{O}/\mathrm{Stab}_G(\mathcal{O}) \simeq B$*,*

and for some value of M

iii) $\mathrm{Stab}_G(\mathcal{O}) \simeq \mathbb{Z}_M$ *if and only if B is a circle, and*

iv) $\mathrm{Stab}_G(\mathcal{O}) \simeq D_M$ *if and only if B is an interval.*

Proof. Statement i): The fixed point set of non-trivial conformal automorphism consists of isolated fixed points.

Statement ii): Let $H = \mathrm{Stab}_G(\mathcal{O})$. There is a natural map $p : \mathcal{O}/H \to S/G^*$ given by $p(Hx) = G^*x$ such that the image $p(\mathcal{O}) = \bar{B}$ as defined in §2. Let $\bar{B}^\circ$ denote $\bar{B}$ with the even periodic points removed. To prove that $\mathcal{O}/H \simeq B$ it suffices to prove that p is 1-1 on the inverse image of $\bar{B}^\circ$. In turn, it suffices to prove that for any two points $x, y \in \mathcal{O}$ and $g \in G^*$ such that $gx = y$ and $|G_y|$ is odd, there is an $h \in \mathrm{Stab}_G(\mathcal{O})$ such that $hx = y$. Let x, y, g be as stated, then $\mathcal{O}$ and $g\mathcal{O}$ are ovals intersecting at y. Let U be a small G_y^* equivariant neighbourhood of y. Since $|G_y|$ is odd, then for any two ovals $\mathcal{O}_1, \mathcal{O}_2$ passing through y there is a $g' \in G_y$ such that $g'(\mathcal{O}_1 \cap U) = (\mathcal{O}_2 \cap U)$ (see Figure 1) and hence $g'\mathcal{O}_1 = \mathcal{O}_2$ by analytic continuation. It follows then that there is a $g' \in G_y$ such that $g'g\mathcal{O} = \mathcal{O}$. If $g'g$ is not orientation preserving then set $h = g'g\theta$, otherwise set $h = g'g$. Then $h \in \mathrm{Stab}_G(\mathcal{O})$) and $hx = y$ as was to be proven.

Statements iii) and iv): First note that an orientation-preserving non-identity homeomorphism of finite order of a circle is fixed-point free. Thus, if H is the subgroup of orientation-preserving elements of $\mathrm{Stab}_G(\mathrm{O})$ (considered as homeomorphisms of the circle) then $\mathcal{O} \to \mathcal{O}/H$ is a covering space and H must be a cyclic group $\mathbb{Z}_M$. If $\mathrm{Stab}(\mathcal{O})$ contains orientation-reversing homeomorphisms (of $\mathcal{O}$!) then they must all be involutions with exactly two fixed points and the product of any two lies in H. It follows then that $\mathrm{Stab}_G(\mathcal{O}) \backsimeq \mathbb{Z}_M$ if and only if B is a circle, and $\mathrm{Stab}_G(\mathcal{O}) \backsimeq D_M$ if and only if B is an interval. ∎

Remark 9 *If L is a line in $\mathbb{H}$ that covers $\mathcal{O}$ then the group of Λ-equivalences* $\mathrm{Stab}_\Lambda(L)$ *will be infinite cyclic if B is a circle and infinite dihedral in case B is a segment. In both cases* $\mathrm{Stab}_G(\mathcal{O}) = \eta(\mathrm{Stab}_\Lambda(L))$. *This will allow us to compute* $\mathrm{Stab}_G(\mathcal{O})$ *in terms of the tiling on $\mathbb{H}$ introduced in §5. Specifically we have in the circle case*

$$\mathrm{Stab}_G(\mathcal{O}) = \langle \eta(\gamma) \rangle, \tag{3.1}$$

where γ is a generator of generator of $\mathrm{Stab}_\Lambda(L)$. *In the segment case let ι_1 and ι_2 be two involutions of L whose fixed points on L are as close as possible. Then* $\mathrm{Stab}_\Lambda(L) = \langle \iota_1, \iota_2 \rangle$ *and hence*

$$\mathrm{Stab}_G(\mathcal{O}) = \langle \eta(\iota_1), \eta(\iota_2) \rangle. \tag{3.2}$$

4 Symmetries and generating vectors

Generating vectors The results of this section are written to allow for the most general possible $\overline{T}$ - since it costs nothing to do so - though we have in mind a disk with a few branch points. The general results are a bit ungainly though they reduce to quite simple equations in the disk case, as illustrated by examples at the end of this section.

We recall the ideas of generating vectors, introduced in [2], which we use as a tool to construct G-actions on surfaces and test whether the surface admits G-symmetries. In particular, we will be able to write down general criteria for the existence of symmetries, normalizing G, in terms of generating vectors. Let us assume that the *branching data* of G is $\mathcal{B} = (\rho : m_1, \ldots, m_r)$, i.e., the genus of T is ρ and the quotient map $S \to S/G = T$ is branched over r points $Q_1, \ldots, Q_r$ with branching orders $m_1, \ldots, m_r$. We say that S admits a G-$(\rho : m_1, \ldots, m_r)$ action. If $\rho = 0$ we just write $\mathcal{B} = (m_1, \ldots, m_r)$ instead of $\mathcal{B} = (\rho : m_1, \ldots, m_r)$.

It is well known that the uniformizing Fuchsian group Λ has a presentation of the following type:

$$\begin{aligned} \Lambda &\simeq \Lambda(\rho : m_1, \ldots, m_r) \qquad (4.1) \\ &= \langle \alpha_i, \beta_i, \gamma_j, 1 \le i \le \rho, 1 \le j \le r : \prod_{i=1}^{\rho} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \rangle, \end{aligned}$$

and that the genus σ of S is given by the Riemann-Hurwitz equation

$$\frac{(2\sigma - 2)}{|G|} = (2\rho - 2 + r) - \sum_{j=1}^{r} \frac{1}{m_j}. \qquad (4.2)$$

Let a_i, b_i, c_j be images of the generators $\alpha_i, \beta_i, \gamma_j$ under the map $\eta : \Lambda \longrightarrow \Lambda/\Gamma \simeq G$ introduced earlier. We get a generating set for G, $\{a_i, b_i, c_j : 1 \le i \le \rho, 1 \le j \le r\}$, satisfying the following properties:

$$\prod_{i=1}^{\rho} [a_i, b_i] \prod_{j=1}^{r} c_j = 1, \qquad (4.3)$$

and

$$o(c_j) = m_j. \qquad (4.4)$$

Each generating set satisfying (4.3) and (4.4) is called a $(\rho : m_1, \ldots, m_r)$ - *generating vector*. There is a 1-1 correspondence between the set of $(\rho : m_1, \ldots, m_r)$

- generating vectors of G and the epimorphisms $\Lambda \to G$, preserving the orders of the γ_i, once a generating set $\mathcal{G} = \{\alpha_i, \beta_i, \gamma_j : 1 \le i \le \rho, 1 \le j \le r\}$ of Λ has been fixed.

Remark 10 *As previously discussed, if ω is an automorphism of G then the epimorphism $\omega \circ \eta : \Lambda \to G$ determines the same surface and an essentially equivalent G-action. Therefore we need only consider $\mathrm{Aut}(G)$-equivalence classes of generating vectors with the obvious action of $\mathrm{Aut}(G)$ on generating vectors. This usually cuts down the number of vectors to a manageable size.*

Markings on T The set $\mathcal{G} = \{\alpha_i, \beta_i, \gamma_j : 1 \le i \le \rho, 1 \le j \le r\}$ can be picked in infinitely many ways, we shall pick one constructed from a marking on T. see [1]. Let $T^\circ = T - \{Q_1, \ldots, Q_r\}$ denote T with the branch points removed, let $q_S : S \to T$ be the quotient map $S \to S/G$, and let $S^\circ = q_S^{-1}(T^\circ)$. Then, $q_S : S^\circ \to T^\circ$ is regular unbranched covering space whose covering group may be identified with G. Correspondingly, let $q_{\mathbb{H}} : \mathbb{H} \to T$ be defined by $\mathbb{H} \to \mathbb{H}/\Lambda$, and $\mathbb{H}^\circ = q_{\mathbb{H}}^{-1}(T^\circ)$. Then, $q_{\mathbb{H}} : \mathbb{H}^\circ \to T^\circ$ is also a regular unbranched covering with Λ as its covering group. Choose base points $x_0 \in \mathbb{H}$, $P_0 \in S$ and $Q_0 \in T$ such that x_0 lies over P_0, which lies over Q_0. It follows then that there are natural maps (see construction below):

$$\xi_G : \pi_1(T^\circ, Q_0) \to G, \quad \xi_\Lambda : \pi_1(T^\circ, Q_0) \to \Lambda \tag{4.5}$$

and the map η may be chosen so that:

$$\xi_G = \eta \circ \xi_\Lambda. \tag{4.6}$$

The group $\pi_1(T^\circ, Q_0)$ has a presentation similar to (4.1):

$$\pi_1(T^\circ, Q_0) \simeq \langle \tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_j : 1 \le i \le \rho, 1 \le j \le r, \prod_{i=1}^{\rho} [\tilde{\alpha}_i, \tilde{\beta}_i] \prod_{j=1}^{r} \tilde{\gamma}_j = 1 \rangle. \tag{4.7}$$

The ordered set of loops $\mathcal{M} = \{\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_j : 1 \le i \le \rho, 1 \le j \le r\}$, is called a $(\rho : m_1, \ldots, m_r)$-*marking* of T. If we set $\mathcal{G} = \{\alpha_i, \beta_i, \gamma_j : 1 \le i \le \rho, 1 \le j \le r\}$, to be the ξ_Λ-images of the generators in the marking, we obtain a generating set for Λ of the type required in (4.1). Some examples of markings with $\rho = 0$, (i.e., T is a sphere, the case of primary interest) are pictured in Figs. 2.a - 2.c and Figures 6.a - 6.b. Observe that the loop $\tilde{\gamma}_j$ encircles a single puncture at a branch point of order m_j.

For use in later sections let us recall the definition of the maps in (4.5) and record some key properties. In the general situation we have a group H acting properly discontinuously (no fixed points) on a space X. Let $q : X \to X/H$ denote the quotient map, and let $x_0 \in X$. There is a homomorphism $\xi_{x_0} : \pi_1(X/H, q(x_0)) \to H$ defined as follows. Let $\alpha \in \pi_1(X/H, q(x_0))$ and let $\tilde{\alpha}$ be the lift of α to X satisfying $\widetilde{\alpha}(0) = x_0$. Then $\tilde{\alpha}(1) = hx_0$ for some $h \in H$, set $\xi_{x_0}(\alpha) = h$. The following statements are easily verified :

$$\xi_{hx_0} = Ad_h \circ \xi_{x_0},\ h \in H. \tag{4.8}$$

More generally if k is a homeomorphism of X covering the homeomorphism $\bar{k}$ on X/H, (so that k normalizes H) and $\bar{k}_*$ denotes the homomorphism

$$\pi_1(X/H, q(x_0)) \to \pi_1(X/H, \bar{k}q(x_0))$$

then

$$\xi_{kx_0} \circ \bar{k}_* = Ad_k \circ \xi_{x_0}. \tag{4.9}$$

Let ζ be a path in X/H with a lift $\tilde{\zeta}$ in X from x_1 to x_0, and let $Ad_\zeta : \pi_1(X/H, q(x_0)) \to \pi_1(X/H, q(x_1))$ be the map $\delta \to \zeta * \delta * \zeta^{-1}$. Then,

$$\xi_{x_1} = \xi_{x_0} \circ Ad_\zeta. \tag{4.10}$$

The maps ξ_G, ξ_Λ, are defined by $\xi_G = \xi_{P_0}, \xi_\Lambda = \xi_{x_0}$ with $P_0 \in S^\circ$ and $x_0 \in \mathbb{H}^\circ$ chosen as above. By (4.8) and (4.10) above, the maps ξ_G, ξ_Λ, and hence η are well-defined up to inner automorphisms.

Existence of symmetries Now suppose that we have chosen a marking based at Q_0, let δ be a path from Q_0 to $\psi(Q_0)$. Then, the map $\Phi = Ad_\delta \circ \psi_* : \alpha \to \delta * \psi_*(\alpha) * \delta^{-1}$ is an automorphism of $\pi_1(T^\circ, Q_0)$. The next lemma follows from (4.9) and (4.10) above.

Lemma 11 *Let θ be a symmetry of S with ovals, normalizing G, let ψ be the induced symmetry of T, and let other notation be as above. Suppose that δ is covered by a path in S from P_0 to $\theta(P_0)$. Then,*

$$\theta\xi_G(\alpha)\theta^{-1} = \xi_G(\Phi(\alpha)) \textit{ for } \alpha \in \pi_1(T^\circ, Q_0). \tag{4.11}$$

Similarly, if $\tilde{\theta}$ is a reflection in $\mathbb{H}$, normalizing Λ and inducing ψ, and δ is covered by a path in $\mathbb{H}$ from x_0 to $\tilde{\theta}(x_0)$ then,

$$\tilde{\theta}\xi_\Lambda(\alpha)\tilde{\theta}^{-1} = \xi_\Lambda(\Phi(\alpha)) \textit{ for } \alpha \in \pi_1(T^\circ, Q_0). \tag{4.12}$$

From this Lemma we may deduce the following theorem which allows us to determine all symmetries normalizing G. The theorem generalizes the result in Singerman [6] which we recover as Corollary 17 below.

Theorem 12 *Let S be a surface with G-action and $T = S/G$. Let ψ be a symmetry of T, with fixed points, preserving the branch set and branching data of the projection $S \to T$. Fix a path δ in T from Q_0 to $\psi(Q_0)$, let Φ be associated automorphism of $\pi_1(T^\circ, Q_0)$, and let all other notation be as above. Then, S admits a symmetry θ, normalizing G and inducing ψ, if and only if there is an automorphism $\Theta \in \mathrm{Aut}(G)$, such that*

$$\Theta(\xi_G(\alpha)) = \xi_G(\Phi(\alpha)) \text{ for } \alpha \in \pi_1(T^\circ, Q_0). \tag{4.13}$$

Proof. Suppose Θ exists. Because ψ has a fixed point and preserves the branch set and the branching data, then ψ lifts to a reflection $\tilde{\psi}$ in $\mathbb{H}$, normalizing Λ. Let δ' be the image of a path in $\mathbb{H}$ from x_0 to $\tilde{\psi}(x_0)$ and not passing through any branch points. The path $\epsilon = \delta' * \delta^{-1}$ lies in $\pi_1(T^\circ, Q_0)$ and satisfies

$$\Theta'(\xi_G(\alpha)) = \xi_G(\Phi'(\alpha)) \text{ for } \alpha \in \pi_1(T^\circ, Q_0),$$

where

$$\Theta' = Ad_{\xi_G(\epsilon)} \circ \Theta,$$

and Φ' is defined with respect to δ'. Also from (4.12) we have:

$$\tilde{\psi}\xi_\Lambda(\alpha)\tilde{\psi}^{-1} = \xi_\Lambda(\Phi'(\alpha)) \text{ for } \alpha \in \pi_1(T^\circ, Q_0),$$

Since $\xi_G = \eta \circ \xi_\Lambda$ we have

$$\eta(\tilde{\psi}\xi_\Lambda(\alpha)\tilde{\psi}^{-1}) = \Theta'(\xi_G(\alpha)), \text{ for } \alpha \in \pi_1(T^\circ, Q_0),$$

and since ξ_Λ is surjective, it follows that $\tilde{\psi}$ normalizes the kernel Γ of $\eta : \Lambda \to G$. It further follows that $\tilde{\psi}$ induces a symmetry θ on $S = \mathbb{H}/\Gamma$ that projects to ψ. This proves sufficiency. To prove necessity suppose that θ exists. Then, by combining several of the equations above and $\eta\tilde{\psi} = \theta\eta$ we get:

$$\theta\xi_G(\alpha)\theta^{-1} = \eta(\tilde{\psi}\xi_\Lambda(\alpha)\tilde{\psi}^{-1}) =$$
$$\Theta'(\xi_G(\alpha)) = \xi_G(\epsilon)\xi_G(\Phi(\alpha))\xi_G(\epsilon^{-1}), \text{ for } \alpha \in \pi_1(T^\circ, Q_0).$$

The required automorphism Θ will be conjugation by $\xi_G(\epsilon^{-1})\theta$. ∎

Corollary 13 *Let all notation be as in Theorem 12, and let $\mathcal{M}$ be a marking on T. For each generator $\alpha \in \mathcal{M}$ let $g_\alpha = \xi_G(\alpha)$ be the associated element of the generating vector and let $h_\alpha = \xi_G(\Phi(\alpha))$. Then S admits a symmetry θ inducing ψ if and only if there is an automorphism Θ of G satisfying.*

$$\Theta(g_\alpha) = h_\alpha, \text{ for } \alpha \in \mathcal{M}. \tag{4.14}$$

Since $\mathcal{M}$ is a generating set the corollary is just a simple restatement of the preceding theorem. The point of the corollary is that the h_α can be easily calculated by substituting the g_α into simple words in the elements of the marking. Furthermore, the words depend only on the topology of T, the location of the branch points, the action of ψ and the selection of the marking. Once these are fixed then the h_α are easily computed for all G and branching orders. Examples are given at the end of the section. Note that if a Θ satisfying (4.14) exists then it is unique

Conjugacy classes of symmetries in G^* As discussed in §3 we need to be able to determine the conjugacy classes of symmetries in G^* and in §2 we saw that the different conjugacy classes correspond to the parts of the dissected boundary. The theorem above allowed us to identify at least one such θ via the induced automorphism Θ but did not allow us to determine the association between the conjugacy classes of symmetries and the parts of the dissected boundary. It turns out we can determine the association by making good choices of the path δ as described in the following paragraph. An alternate method is described in Remark 26 of §5 when $\overline{T}$ is a disc.

Let $B_0, \ldots, B_k$ be some ordering of the parts of the dissected boundary. For each B_i let R_i be a non-periodic point of B_i. Construct δ_i as follows. Let ν_i be a path from Q_0 to R_i not passing through any points of the fixed point subset except possibly at the endpoints, or unless $Q_0 = R_0$ in which case ν_0 is the constant path at Q_0. The path $\psi_*(\nu_i^{-1})$ is the mirror image starting at R_i and moving to $\psi(Q_0)$. Let $\delta_i = \nu_i * \psi_*(\nu_i^{-1})$.

Proposition 14 *Let S be a surface with G-action, admitting symmetries with ovals that normalize G. Select a marking on T and let δ_i be defined as above. Let Φ_i be the automorphism of $\pi_1(T^\circ, Q_0)$ defined by $\Phi_i : \alpha \to \delta_i * \psi_*(\alpha) * \delta_i^{-1}$. Then, there are symmetries $\theta_0, \ldots, \theta_k$ such that the corresponding automorphisms Θ_i induced the θ_i satisfy the following:*

i) $\Theta_i(\xi_G(\alpha)) = \xi_G(\Phi_i(\alpha))$, *for* $\alpha \in \pi_1(T^\circ, Q_0)$, *ii)* θ_i *is associated to* B_i*, and*

iii) $\theta_i = \theta_0 d_i$ *where*

$$d_i = \xi_G(\delta_0 * \delta_i^{-1})$$

and

$$\theta_0 d_i \theta_0 = d_i^{-1}.$$

Proof. Note that if Θ exists as in (4.13) then Θ_i exists and is given by:

$$\Theta_i(g) = \xi_G(\delta_i * \delta^{-1})\Theta(g)\xi_G(\delta * \delta_i^{-1}),\ g \in G. \tag{4.15}$$

Now fix i and let y_i be the endpoint of the lift of ν_i^{-1} to S starting at P_0. Since y_i projects to R_i there is a unique θ_i whose fixed point set passes through y_i. Clearly θ_i is associated with B_i and induces the map ψ on T. We obtain for $\alpha \in \pi_1(T^\circ, Q_0)$

$$\begin{aligned}
\Theta_i(\xi_G(\alpha)) &= \xi_{P_0}(\nu_i * \psi_*(\nu_i^{-1}) * \psi_*(\alpha) * \psi_*(\nu_i) * \nu_i^{-1}) \\
&= \xi_{y_i}(\psi_*(\nu_i^{-1}) * \psi_*(\alpha) * \psi_*(\nu_i)) \quad \text{(by (4.10))} \\
&= \xi_{y_i}(\psi_*(\nu_i^{-1} * \alpha * \nu_i)) \\
&= \theta_i \xi_{y_i}(\nu_i^{-1} * \alpha * \nu_i)\theta_i^{-1} \quad \text{(by (4.9)} \\
&= \theta_i \xi_{P_0}(\alpha)\theta_i^{-1}. \quad \text{(by (4.10))}
\end{aligned}$$

This proves *i*) and *ii*).

To prove *iii*) let us define d_i by $d_i = \theta_0\theta_i$ so that $\theta_i = \theta_0 d_i$. It suffices to prove that the lift $\delta_0 * \delta_i^{-1}$ is $\nu_0 * \psi_*(\nu_0^{-1}) * \psi_*(\nu_i) * \nu_i^{-1}$. By construction and definition $\nu_0 * \psi_*(\nu_0^{-1})$ has a lift in the following two pieces: the lift of ν_0 from Q_0 to y_0 followed by the lift of $\psi_*(\nu_0^{-1})$ from y_0 to $\theta_0 P_0$, since $\theta_0(\tilde{\nu}_0^{-1})$ projects to $\psi_*(\nu_0^{-1})$. Similarly $\psi_*(\nu_i) * \nu_i^{-1}$ has a lift in these two pieces: the lift of $\psi_*(\nu_i)$ from $\theta_i P_0$ to y_i followed by the lift of ν_i^{-1} from y_i to P_0. Now if we translate the lift of $\psi_*(\nu_i) * \nu_i^{-1}$ by d_i, then this translated lift and the lift of $\nu_0 * \psi_*(\nu_0^{-1})$ match at the point $\theta_0 P_0 = d_i\theta_i P_0$ and hence the lift $\delta_0 * \delta_i^{-1}$ starts at P_0 and finishes at $d_i P_0$. All is now proven. ■

Remark 15 *Of course this proposition may be applied by writing* $\delta_0 * \delta_i^{-1}$ *in terms of the generating loops and then computing their* ξ_G*-images by means of the generating vector.*

Computing Φ and Θ We can simplify the application of the existence criterion if T is a sphere and ψ has fixed points. First of all the G-action on S is given (up to equivalence) by one of a (usually small) number of generating vectors. To see if a symmetry exists we merely need to find an automorphism Θ that satisfies (4.13) for the generating loops in the marking and their Φ-images as proven in Corollary 13. Once we have discovered the elements g_α and h_α of the Corollary we simply check (4.14) against all automorphisms Θ of order 2. For small groups and plenty of large ones this is easily programmed in the packages such as GAP [8] or MAGMA [9].

Since we assumed T is a sphere and ψ has an oval, we may assume that ψ is reflection in some line through the origin, e.g., complex conjugation. In this case the oval is the real line, compactified into a circle by adding the point at infinity. We make take the compactified upper half plane, a disc, as a model of $\overline{T}$. The symmetric branch points are real and the involutary pairs are complex conjugate pairs, one point of each involutary pair lies in the interior of $\overline{T}$. Now select a marking on T, there will be many choices. It is obvious that we will achieve a good deal of simplification in computing Φ if we choose a marking which is well adapted to ψ. For example, we may select the marking as in the steps below (see Figures 2.a - 2.c and Figures 6.a - 6.b):

- Select a distinguished part B_0 of the dissected boundary $\partial(\overline{T})$.
- Pick Q_0 to be a non-periodic point of B_0, so that the loop δ_0 is trivial.
- Each loop in the marking should consist a path ζ_i nearly to the branch point Q_i, a counter-clockwise circle κ_i about the branch point and the same path back to the base point, i.e., $\tilde{\gamma}_i = \zeta_i * \kappa_i * \zeta_i^{-1}$. For symmetric points the path should attach to the loop at a point of intersection of the circle and the fixed point set of ψ. For involutary pairs the loops should be ψ-images of each other. If ∞ is one of the branch points then select for the corresponding κ_i a large clockwise circle outside all the other branch points.
- If the loops are labelled $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ as we move counterclockwise around the base point then $\tilde{\gamma}_1 \cdots \tilde{\gamma}_r = 1$ holds.
- For an involutary pair $\{Q_i, Q_j\}$, $\psi_*(\gamma_i) = \gamma_j^{-1}$ and $\psi_*(\gamma_j) = \gamma_i^{-1}$.
- For a symmetric point Q_i we have $\psi_*(\gamma_i) = \epsilon_i * \gamma_i^{-1} * \epsilon_i^{-1}$, where $\epsilon_i = \psi_*(\zeta_i) * \zeta_i^{-1}$. The latter curve is easily determined.

Example 16 *Suppose that T is a sphere with 3 branch points. As previously noted in Example 1, we may assume that the three branch points are $Q_1 = -1$, $Q_2 = 1$, and $Q_3 = \infty$, that we have chosen $Q_0 = 0$, and that symmetries are either $\psi^1 : z \to \bar{z}$ or $\psi^2 : z \to -\bar{z}$. In the first case all the branch points are fixed, so m_1, m_2 and m_3 can be arbitrary, in the second case we must have $m_1 = m_2$. In both cases we may select the marking so that the arcs ζ_1 and ζ_2 move along the real axis away from 0 and ϵ_3 moves away from zero along the positive imaginary axis. It is easy to see that the following hold:*

$$\psi^1_*(\gamma_1) = \gamma_1^{-1},\ \psi^1_*(\gamma_2) = \gamma_2^{-1}$$

and

$$\psi^2_*(\gamma_1) = \gamma_2^{-1},\ \psi^2_*(\gamma_2) = \gamma_1^{-1}.$$

Since $\gamma_3 = \gamma_2^{-1}\gamma_1^{-1}$ its images is determined by the two equations above. We recover theorem in Singerman's paper [6].

Corollary 17 *Let S be a surface with a G-(m_1, m_2, m_3) action defined by the generating vector (a, b, c). Then the G action extends to a symmetric G-(m_1, m_2, m_3) action (i.e., admits a symmetry θ normalizing the G-action) if and only if there is an automorphism Θ of G that satisfies either*

$$\Theta(a) = a^{-1},\ \Theta(b) = b^{-1}, \tag{4.16}$$

or

$$\Theta(a) = b^{-1},\ \Theta(b) = a^{-1}. \tag{4.17}$$

In the second case we must have $m_1 = m_2$. The automorphism Θ is induced by the symmetry θ.

On the other hand suppose that we start with a group G satisfying:

$$G = < a, b >,\ o(a) = m_{1,}\ o(b) = m_{2,}\ o(ab) = m_3,$$

and that there is an automorphism Θ satisfying one of the above equations, and that

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right)$$

is an integer. Then there is a symmetric G-(m_1, m_2, m_3) action on a surface S of genus σ such that there is a symmetry θ inducing Θ.

Remark 18 *If S has a tiling by polygons such that the reflections in the sides of polygons are symmetries then relevant cases are where all branch points are on the mirror of ψ or there is at most one involutary pair. For tilings by triangles equation (4.16) is the relevant case from Corollary 17.*

Genus 3 example with $G = \mathbb{Z}_3$. To keep things simple we will look at the action of the small group $\mathbb{Z}_3$ where the quotient T has genus zero. According to [2] the branching data must be $(3^5) = (3, 3, 3, 3, 3)$. The set of possible generating vectors up to $\mathrm{Aut}(G)$-equivalence are easily calculated:

$$\begin{aligned}\mathbf{v}_1 &= (-1, 1, 1, 1, 1),\\ \mathbf{v}_2 &= (1, -1, 1, 1, 1),\\ \mathbf{v}_3 &= (1, 1, -1, 1, 1),\\ \mathbf{v}_4 &= (1, 1, 1, -1, 1),\\ \mathbf{v}_5 &= (1, 1, 1, 1, -1).\end{aligned}$$

There are three different cases for the geometry of T:

Case 5-0:	five symmetric points,
Case 3-2:	three symmetric points and one involutary pair,
Case 1-4:	one symmetric point and two involutary pairs.

These possibilities, along with a selection of the markings for T° have been illustrated in Figures 2.a, 2.b and 2.c. In the figures the fixed point set of ψ is the dotted horizontal line.

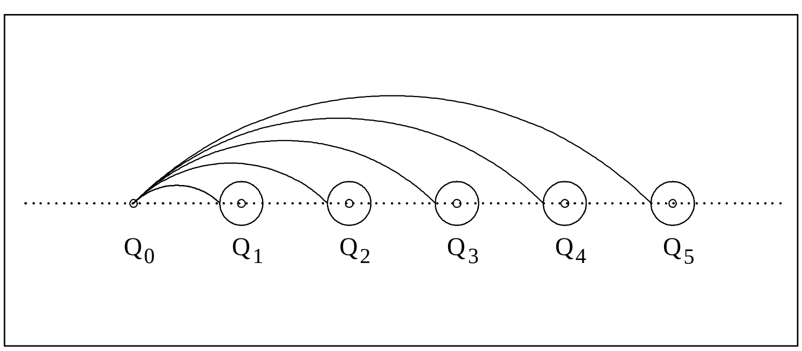

Figure 2.a. Case 5-0

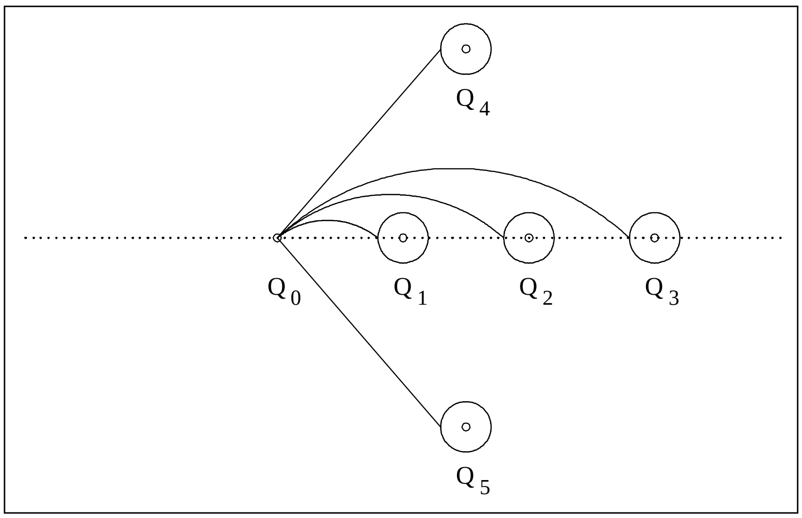

Fig 2.b. Case 3-2

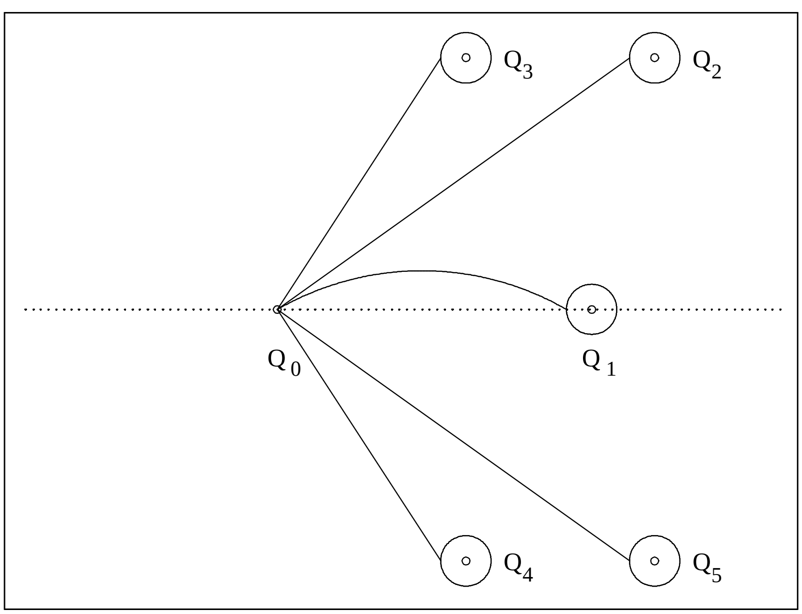

Fig 2.c. Case 1.4

With the labelling given, we have $\tilde{\gamma}_1 \cdots \tilde{\gamma}_5 = 1$. By looking at "before and after" ψ-pictures we get the following formulae in the three different cases. Also note that the formulas can be simplified by using the generating relations.

Case 5-0	Case 3-2	Case 1-4
$\tilde{\gamma}_1 \to \tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_1 \to \tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_1 \to \tilde{\gamma}_1^{-1}$
$\tilde{\gamma}_2 \to \tilde{\gamma}_1\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_2 \to \tilde{\gamma}_1\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_2 \to \tilde{\gamma}_5^{-1}$
$\tilde{\gamma}_3 \to \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3^{-1}\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_3 \to \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3^{-1}\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_3 \to \tilde{\gamma}_4^{-1}$
$\tilde{\gamma}_4 \to \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4^{-1}\tilde{\gamma}_3^{-1}\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\gamma_4 \to \tilde{\gamma}_5^{-1}$	$\tilde{\gamma}_4 \to \tilde{\gamma}_3^{-1}$
$\tilde{\gamma}_5 \to \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4\tilde{\gamma}_5^{-1}\tilde{\gamma}_4^{-1}\tilde{\gamma}_3^{-1}\tilde{\gamma}_2^{-1}\tilde{\gamma}_1^{-1}$	$\tilde{\gamma}_5 \to \tilde{\gamma}_4^{-1}$	$\tilde{\gamma}_5 \to \tilde{\gamma}_2^{-1}$

Observe that in the above table every $\tilde{\gamma}_j$ is taken to a conjugate of $\tilde{\gamma}_{j'}^{-1}$ for some permutation $j \to j'$ of the indices. Since G is abelian the induced map on G will map c_j to $c_{j'}^{-1}$ for the same permutation of the indices. We get:

Case 5-0	Case 3-2	Case 1-4
$\Theta(c_1) = c_1^{-1}$	$\Theta(c_1) = c_1^{-1}$	$\Theta(c_1) = c_1^{-1}$
$\Theta(c_2) = c_2^{-1}$	$\Theta(c_2) = c_2^{-1}$	$\Theta(c_2) = c_5^{-1}$
$\Theta(c_3) = c_3^{-1}$	$\Theta(c_3) = c_4^{-1}$	$\Theta(c_4) = c_4^{-1}$
$\Theta(c_4) = c_4^{-1}$	$\Theta(c_4) = c_5^{-1}$	$\Theta(c_4) = c_3^{-1}$
$\Theta(c_5) = c_5^{-1}$	$\Theta(c_5) = c_4^{-1}$	$\Theta(c_5) = c_2^{-1}$

Since there are an odd number of branch points then one of the indices must be fixed and it follows that a generator of the cyclic group G is mapped to its inverse. Thus the map Θ is $g \to g^{-1}$. If we identify the $\mathrm{Aut}(G)$-equivalence classes with the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$ then we get the following equivalence classes that satisfy the symmetry equations (4.14).

Case 5-0	Case 3-2	Case 4-1
$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$\mathbf{v}_1$

(4.18)

We can also see in this example what restrictions are imposed on the branch points. For example in the Case 5-0 all the branch points must lie on the fixed point set of a reflection on the sphere, i.e., they must all lie on some circle. We can also see that some configurations of branch points allow more than one possible symmetry. For instance, if the branch points are $\{-2, -1, 0, 1, 2\}$ then symmetries of types 5-0 and 1-4 are both possible. In this case the group G will not equal $\mathrm{Aut}(S)$.

5 The tiling of $\mathbb{H}$ induced by a symmetry

In this section we shall introduce the tiling induced by a symmetry, which we will use in §6 to compute the order of stabilizers of ovals. Construct the following subsets of $\mathbb{H}$. Let V be the set of all periodic points lying on the boundary $\partial(\overline{T})$ of $\overline{T}$. Let $q_{\mathbb{H}} : \mathbb{H} \to \overline{T} \simeq \mathbb{H}/\Lambda^*$ denote the quotient projection and define $V_{\mathbb{H}} = q_{\mathbb{H}}^{-1}(V)$,and $E_{\mathbb{H}} = q_{\mathbb{H}}^{-1}(\partial(\overline{T}))$. The set $E_{\mathbb{H}}$ is a locally finite union of closed hyperbolic segments and lines in $\mathbb{H}$. The set $\mathbb{H} - E_{\mathbb{H}}$ is a disjoint union

$$\mathbb{H} - E_{\mathbb{H}} = \bigcup_i U_i^\circ$$

of path-connected open subsets of $\mathbb{H}$ each of whose boundaries is a union of segments and lines from $E_{\mathbb{H}}$. Let U_i be the closure of U_i°. Each U_i is called a *tile*, and the collection of tiles $\{U_i\}$ is called a *tiling* of $\mathbb{H}$. The set $E_{\mathbb{H}} - V_{\mathbb{H}}$ is a disjoint locally finite union of lines and open hyperbolic segments. The closure of a segment or line in each $E_{\mathbb{H}} - V_{\mathbb{H}}$ will be called an edge of the tiling. An edge which is a line will be called a *line-edge* otherwise it will be called a *segment-edge.* The points in $V_{\mathbb{H}}$ are called the vertices of the tiling. The path components of the boundary $\partial(U)$ of a tile U are called the *boundary components* of U. Finally, we shall select a distinguished tile U_0 from the tiling which we call the *master tile*.

It is easily seen that the following hold.

(5.1) For each segment $s \subset E_{\mathbb{H}}$ the hyperbolic line determined by s is contained in $E_{\mathbb{H}}$, and this line is the fixed line of some reflection in Λ^*.

(5.2) Each point in $\mathbb{H}$ has a neighbourhood which meets $E_{\mathbb{H}}$ in an empty set, a segment, or segments, all intersecting at a vertex, and such that the angles between adjacent segments are equal (see Figure 1 for an example).

(5.3) Two edges of the tiling are Λ-equivalent if and only if they have the same image in $\partial(\overline{T})$. The image will be a circle with at most one periodic point or an arc with periodic endpoints and no periodic interior points.

Each line determined as in (5.1) above is called a *line of the tiling.* The path components of the boundary $\partial(U)$ of a tile U are called the *boundary components* of U.

Remark 19 *The tiles are polygons if and only if $\overline{T}$ is a disc with at most one interior branch point. This fact is not needed for our development so we leave* *the proof to the reader. In case the boundary is not finite there are infinitely many boundary components.*

The tiling has several important properties which we list in the following proposition.

Proposition 20 *Let the tiling $\{U_i\}$ of $\mathbb{H}$, be as defined above. Then we have the following.*

i) Each tile is a convex closed subset of $\mathbb{H}$.

ii) Two different tiles are disjoint, meet in a vertex or meet along an edge of the tiling.

iii) A boundary component of a tile is homeomorphic to either a circle or a line.

iv) Each element of Λ^* maps tiles onto tiles, and Λ^* permutes the tiles transitively.

v) For each point $x \in \mathbb{H}$ the isotropy subgroup Λ^*_x permutes simply transitively the tiles containing x, unless x is an interior periodic point.

Proof. Statement *i*): Select a tile U and let V be any other tile. Let $u \in U^\circ$ and $v \in V^\circ$ be interior points of the tiles. Since there are only a countable number of vertices we may assume, by adjusting the points u, v if necessary, that the hyperbolic line segment $\overline{uv}$ from u to v does not pass through any vertex. This segment meets at most a finite number of edges in $E_{\mathbb{H}}$. Consider the line L determined by the first segment of $E_{\mathbb{H}}$ encountered while travelling along $\overline{uv}$ from u to v. Since L separates $\mathbb{H}$ into two parts, it separates the interiors of U and V. It follows then that the tile interior U° (tile U) is the intersection of a family of open half planes (closed half planes).

Statement *ii*): The intersection of two tiles is a closed convex set which is contained in a union of line-edges or segment-edges. Thus it must be a point, a line-edge or a sequence of segment-edges. By looking at the local picture at a vertex, it is obvious that two tiles cannot meet along two consecutive segment-edges in a line of the tiling.

Statement *iii*): We observe that a boundary component is either a line or a union of segments. The case of a line is trivial, for the case of segments if suffices to observe that each segment in the boundary meets exactly one other segment in the boundary of the tile in a common endpoint.

Statement *iv*): Observe that, by definition, $E_{\mathbb{H}}$ is Λ^*-equivariant, hence each element of Λ^* maps the interiors of the tiles to interiors of tiles. Since Λ^* contains the inverse of each of its elements, each such transformation maps the interior of a tile bijectively to the interior of another tile. By taking limits, and considering inverses it follows that each element of Λ^* maps each tile bijectively to another tile. To see that Λ^* permutes the tiles transitively, argue as follows. Two tiles which meet along an edge are mapped into each other by the reflection in the common edge. Now consider the tiles U and V and the line segment $\overline{uv}$ constructed in the proof of *i*). Using this line segment, it is easy to produce a sequence of tiles $U = U_0, \ldots, U_t = V$ such that the pair of tiles U_i and U_{i+1} have a common edge for $i = 0, ..., n-1$. It follows that U and V are Λ^*-equivalent.

Statement *v*): This follows directly from (5.2). ■

Tile self-equivalences and boundary shifts Unless $\overline{T}$ is a disc without any interior periodic points then there will be non-identity elements of Λ^* that map a tile to itself. Let $\Lambda^*_U = \mathrm{Stab}_{\Lambda^*}(U)$ be this group of *tile self-equivalences.* We shall be particularly interested in tile self-equivalences that map a given boundary component into itself. Such a tile self-equivalence is called a *boundary shift.* The following result describes some properties of the group of tile self-equivalences.

Lemma 21 *Let $x, y \in U$ be two points of a tile U which are Λ^*-equivalent. Then, the following hold.*

i) There is an element $g \in \Lambda^*_U$ such that $y = gx$.

ii) Λ^*_U contains no reflections.

iii) If x and y are distinct and lie on the same boundary component then the boundary component is invariant under g, i.e., g is a boundary shift, and g is fixed point free when restricted to the boundary component.

Proof. *i*) Let $g \in \Lambda^*$ satisfy $gx = y$. If $gU \neq U$, then gU is a tile meeting U at y. According to *v*) of Proposition 20 there is an $h \in \Lambda^*_y$ such that $hgU = U$.

ii) Now assume that the g above satisfies $gU = U$. If g is a reflection then the reflecting line of g is the perpendicular bisector of the line segment $\overline{xy}$. Because

U is convex the interior of the line segment $\overline{xy}$ crosses the interior of U or lies in the boundary of U. In either case the fixed line of g will contain an interior point of U, a contradiction.

iii) Now suppose that x and y lie on the same boundary component of U. Let α be the path from x to y contained in the boundary component. The translates $g^n\alpha$, $n \in \mathbb{Z}$ may be placed end to end to form an open and closed, path-connected subset of the ∂U and hence equals the original boundary component. If g has a fixed point on the boundary component then g must be a rotation by *ii)*, but, then g cannot preserve the tile. ■

Proposition 22 *Let $q_{\mathbb{H}} : \mathbb{H} \to \overline{T}$, $\mathbb{H} \to \mathbb{H}/\Lambda^*$ be the quotient map, let U be a tile and all other notation be as above. Then, the following hold.*

i) The restricted map $q_{\mathbb{H}} : U \to \overline{T}$ is a branched covering, branched only over the interior periodic points of $\overline{T}$.

ii) The group of covering transformations of the covering in i) is Λ^*_U, i.e., $U/\Lambda^*_U = \overline{T}$. The surface $\overline{T}$ is orientable if and only if $\Lambda^*_U \subseteq \Lambda$.

iii) If E is a boundary component of U then $C = q_{\mathbb{H}}(E)$ is a boundary component of $\overline{T}$, and the covering $q_{\mathbb{H}} : E \to C$ is equivalent to a covering of the circle. The group of boundary shifts of E is the covering group of this restricted projection.

Proof. *i)* and *ii)* By *iv)* of Proposition 20, $q_{\mathbb{H}} : U \to \overline{T}$ is surjective, and by *i)* of Lemma 21 this map is equivalent to the projection $U \to U/\Lambda^*_U$. If $x \in U$ is not a periodic interior point and then x has a small neighbourhood V in U, in the shape of a sector, such that $q_{\mathbb{H}}^{-1}(q_{\mathbb{H}}(V))$ is the union

$$\bigcup_{g \in \Lambda^*_U} gV.$$

and so $V \to q(V)$ is a bijection onto the neighbourhood $q(V)$ of $q(x)$.

iii) This follows from *i)*,the description of boundary components and *iii)* of Lemma 21. ■

Boundary shifts, reflection products and generating vectors In order to use the tiling to compute $\mathrm{Stab}_G(\mathcal{O})$ in terms of the generating vectors we need to be able to calculate boundary shifts and products of reflections in terms of the generating set $\mathcal{G} = \{\alpha_i, \beta_i, \gamma_j\}$. We do this in the next couple of propositions, after introducing some notation. Let U be a tile and e an edge of the tiling. For any interior point on the edge there is a unique direction vector $\mathbf{n}$, normal to the edge and pointing into the interior of the tile. If $\mathbf{v}$ is one of the two directions along the edge then we say that $\mathbf{v}$ points in the *positive direction* along the edge if $\mathbf{n}$ is 90° clockwise from $\mathbf{v}$. Observe that each boundary component can be given a positive orientation that is consistent with all the orientations of its edges. Every component of the boundary $\partial(\overline{T})$ has a neighbourhood collar which is orientable. If we select an orientation on a collar an orientation is induced on the corresponding boundary component of $\overline{T}$.

Select a boundary component of C of $\partial\overline{T}$, a master tile U_0, a boundary component E of U_0, and an orientation of a collar of C such that the following holds for the composite map $U_0 \to T \to \overline{T}$.

(5.4) E maps to C in an orientation preserving manner, and

(5.5) the point x_0 from which ξ_Λ is formed lies in U_0. Recall that the image of x_0 in T is Q_0.

We are going to construct some points and curves in $\mathbb{H}$ and T which are illustrated in Figures 3 and 4. In Figure 3 we have drawn a portion of a master tile in the disc model of the hyperbolic plane. The large dotted circle is the disc boundary, the dotted arcs of circles are sides of the master tile making up a portion of E, and the remaining loops and curves are used to construct elements of the fundamental group. In Figure 4 we are assuming that T is a sphere and that the large dotted circle represents C, the fixed circle of ψ. We may identify $\overline{T}$ with the disc enclosed by this circle and so we see all of $\overline{T}$ in Figure 4 plus a portion of the complement of $\overline{T}$ in T. The other loops and curves in Figure 4 are images of loops and curves in Figure 3.

If C has no periodic points then E is a line, we skip this easy case. Suppose then that C has s periodic points which breaks up C into s intervals ($s = 3$ in Figures 3 and 4). Then the component E either consists of a doubly infinite sequence of edges, $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$, or a cyclic sequence of edges, $\{e_0, e_1, \ldots, e_{sn}\}$. In the finite case U_0 is a polygon with a unique branch point of order n at the centre. Thus $\overline{T}$ is a disc, also with a unique periodic point of order n in the interior of $\overline{T}$. By defining $e_{i+sn} = e_{i-sn} = e_i$ may describe both cases by a doubly infinite sequence. By construction, the edges e_i and e_{i+ks} project to the same interval in C for all i and k, and, s is the smallest integer for which this is true. We assume that as we move along the edges in increasing order we also move along E in the positive direction. Let τ be the generator of the boundary shift group of E that maps e_i to e_{i+s}, and let r_i denote the reflection in e_i.

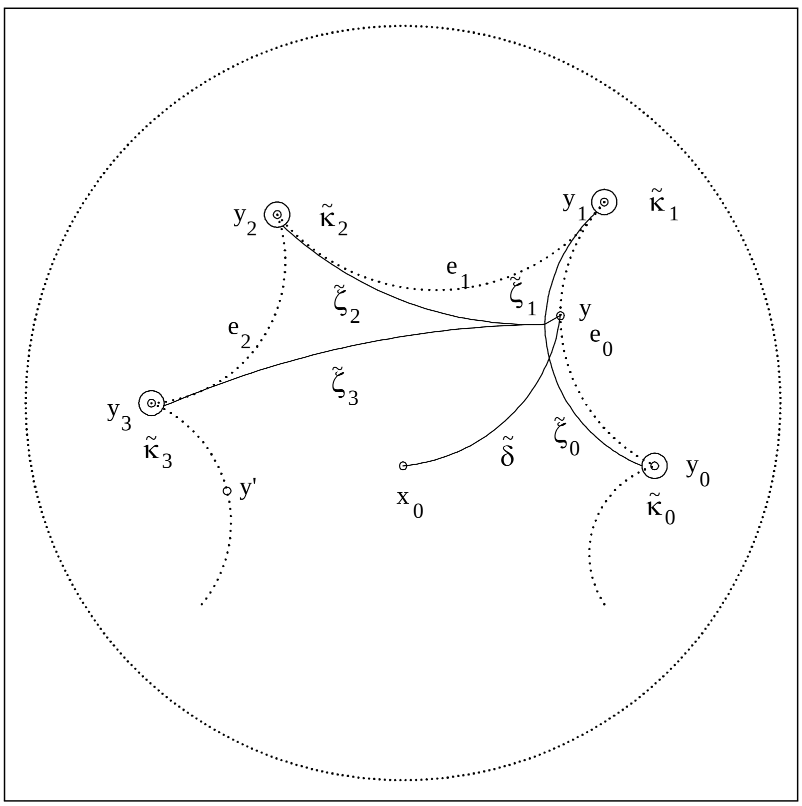

Figure 3. Paths in the master tile U_0 and $\mathbb{H}$

Construct the following points and curves in U_0 and $\mathbb{H}$ (Figure 3) and their images in T (Figure 4). Let y_i be the common point of intersection of e_{i-1} and e_i, so that $e_i = \overline{y_i y_{i+1}}$, and let y be some interior point on e_0. Let f_i, z_i and z be the images of e_i, y_i and y in T, respectively. Let $\tilde{\delta}$ be a curve in U_0 from x_0 to y, which meets ∂U_0 only at its endpoints. For $i > 0$, let $\tilde{\zeta}_i$ be a curve that starts at y, moves into the interior of U_0 a short distance and then runs almost parallel to E, in the positive direction until it gets near y_i. In the finite case it runs parallel to a portion of e_0, then runs parallel to $e_1, e_2, \ldots, e_i$, winding around the boundary of the polygon several times if necessary. For $i \leq 0$, $\tilde{\zeta}_i$ is similarly defined but runs in the negative direction along E. (To keep the picture in Figure 3 uncluttered the curves representing $\tilde{\zeta}_2$ and $\tilde{\zeta}_3$ have not been shown running parallel to the boundary through their entire length, instead they "cut corners". The pictured curves however are obviously homotopic to $\tilde{\zeta}_2$ and $\tilde{\zeta}_3$.) Let $\widetilde{\kappa}_i$ be a small circle centred at y_i and passing through the endpoint of $\tilde{\zeta}_i$. Let δ and ζ_i be the images in T of $\tilde{\delta}$ and $\tilde{\zeta}_i$, respectively. The curve ζ_i starts out from z, runs nearly parallel to C, possibly for several circuits, and ends near z_i. Let κ_i be a small, clockwise circular loop around y_i, starting and finishing at the endpoint of ζ_i. We also assume that κ_i is chosen so that $\widetilde{\kappa}_i$ is mapped to κ_i as an m_i-fold cover where m_i is the branching order at y_i. Finally let $\epsilon_i = \zeta_i * \kappa_i * \zeta_i^{-1}$, and let ϵ (not pictured) be a curve that starts at z, runs out from C for a short distance and then runs parallel to C in the positive direction for one full circuit and then heads back to C ending at z.

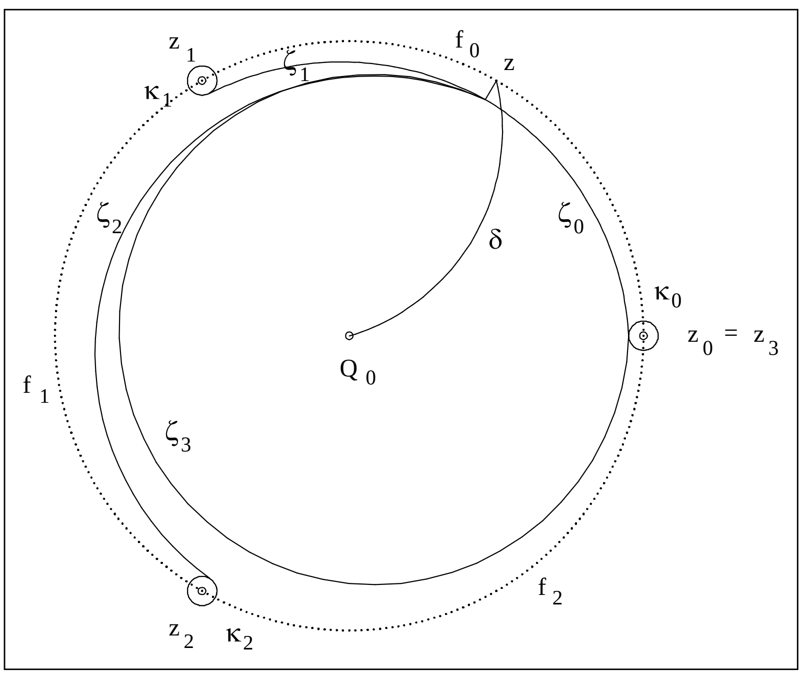

Fig 4. Paths on T

Proposition 23 *Let all curves be defined as above. Then,* *i) The element* $\tau = \xi_\Lambda(\delta * \epsilon * \delta^{-1})$ *is a boundary shift in the positive direction along* E *mapping* e_i *onto* e_{i+s}.

ii) Let r_i *be the reflection in* e_i*, and* $\upsilon_i = r_i r_{i-1}$*, then*

$$\begin{aligned} r_i r_{i-1} &= \xi_\Lambda(\delta * \epsilon_i * \delta^{-1}) \\ \tau r_i \tau^{-1} &= r_{i+s}, \\ \tau \upsilon_i \tau^{-1} &= \upsilon_{i+s}. \end{aligned} \tag{5.6}$$

Proof. *i)* Let $\gamma \in \Lambda$ be the boundary shift which shifts E in the positive direction and takes e_0 onto e_s. The covering curve $\tilde{\epsilon}$ of ϵ starting at y, moves from y into the interior of U_0 for a small distance, then moves parallel to E until it gets near the point $y' = \gamma y$ on e_s, and then heads back to e_s ending at y' (see Figure 3 for location of y'). Now obviously $\xi_x(\epsilon) = \gamma$, and so $\tau = \xi_\Lambda(\delta * \epsilon * \delta^{-1}) = \gamma$, from (4.9) and τ is the claimed boundary shift.

ii) Let w_i be the endpoint of the lift of ζ_i. The lift of κ_i starting at w_i must be the arc on $\widetilde{\kappa}_i$ with one endpoint at w_i and subtending a clockwise angle of $\frac{2\pi}{m_i}$. Since $r_i r_{i-1}$ is a counter-clockwise rotation centred at y_i through an angle of $\frac{2\pi}{m_i}$ it follows that $\xi_{w_i}(\kappa_i) = r_i r_{i-1}$. Again using (4.9) we see that

$$r_i r_{i-1} = \xi_{w_i}(\kappa_i) = \xi_\Lambda(\delta * \zeta_i * \kappa_i * \zeta_i^{-1} * \delta^{-1}) = \xi_\Lambda(\delta * \epsilon_i * \delta^{-1}).$$

■

Remark 24 *The difficulty in using the previous proposition is that it is not always possible to easily determine the curve* δ *if we pick* E *first. However, by picking a* δ *first the calculation is much easier. Let* δ *be a conveniently chosen curve in* T *from* Q_0 *to the point* z *on* $\partial(\overline{T})$*, such that it meets* $\partial(\overline{T})$ *only at its endpoints. The lift of* δ *will be a path in* U_0 *to a point on a boundary component* E' *of* U_0 *which will be* Λ*-conjugate to* E*. Now we may compute* τ' *and the* υ_i' *via the easily identified elements* $\delta * \epsilon * (\delta)^{-1}$ *and* $\delta * \epsilon_i * \delta^{-1}$ *of* $\pi(T^\circ, Q_0)$*. For our calculations in the next section this will be sufficient.*

Remark 25 *For calculations in* G *let*

$$t = \eta(\tau), \quad u_i = \eta(\upsilon_i).$$

If there is more than one boundary component in $\overline{T}$ *a separate notation and calculation will be required for each component. Also note that we need only calculate* τ *and* $\upsilon_1, \ldots, \upsilon_s$ *by virtue of equation (5.6).*

Remark 26 *In §4 we showed how to compute representatives of the different symmetry classes in G^*. Alternatively we may use the results of this section to compute the images of the r_i's in G^* and use one representative symmetry $\eta(r_i)$ for each B in the dissected boundary. As before, write $G^* = \langle\theta_0\rangle \ltimes G$, where we may assume that $\theta_0 = \eta(r_0)$ Since $u_i = \eta(r_i r_{i-1})$ then $\eta(r_i)$, the symmetry induced by r_i is $u_i u_{i-1} \ldots u_1 \theta_0 = \theta_0 \Theta_0(u_i u_{i-1} \ldots u_1)$. By selecting appropriately from the list of the $\eta(r_i)$ we can find a set of symmetry representatives.*

More on the genus 3 example with $G = \mathbb{Z}_3$ The above calculations can be readily carried out for the genus 3 examples considered in the previous section, by considering the markings in Figs. 2.a, 2.b, 2.c. In this example δ can be chosen to be trivial. The elements ϵ and ϵ_i can then be determined for later use in §6.

Case 5-0	$\epsilon = 1$	$\epsilon_1 = \tilde{\gamma}_1^{-1}, \epsilon_2 = \tilde{\gamma}_2^{-1}, \epsilon_3 = \tilde{\gamma}_3^{-1}, \epsilon_4 = \tilde{\gamma}_4^{-1}, \epsilon_5 = \tilde{\gamma}_5^{-1}$
Case 3-2	$\epsilon = \tilde{\gamma}_4$	$\epsilon_1 = \tilde{\gamma}_1^{-1}, \epsilon_2 = \tilde{\gamma}_2^{-1}, \epsilon_3 = \tilde{\gamma}_3^{-1}$
Case 1-4	$\epsilon = \tilde{\gamma}_2\tilde{\gamma}_3$	$\epsilon_1 = \tilde{\gamma}_1^{-1}$

In terms of the τ and υ notation above and the generating set $\{\gamma_1, \gamma_2, \gamma_3, \gamma_5, \gamma_5\}$ we get:

Case 5-0	$\tau = 1$	$\upsilon_1 = \gamma_1^{-1}, \upsilon_2 = \gamma_2^{-1}, \upsilon_3 = \gamma_3^{-1}, \upsilon_4 = \gamma_4^{-1}, \upsilon_5 = \gamma_5^{-1}$
Case 3-2	$\tau = \gamma_4$	$\upsilon_1 = \gamma_1^{-1}, \upsilon_2 = \gamma_2^{-1}, \upsilon_3 = \gamma_3^{-1}$
Case 1-4	$\tau = \gamma_2\gamma_3$	$\upsilon_1 = \gamma_1^{-1}$

The values of t and the u are obtained by replacing γ_i by c_i in the appropriate formulas.

6 Computing stabilizers of ovals

In this section we show how to compute the order of $\mathrm{Stab}_G(\mathcal{O})$ by using the tiling on $\mathbb{H}$ to compute the generators of $\mathrm{Stab}_\Lambda(L)$ and then using (3.1) and (3.2) to calculate $\mathrm{Stab}_G(\mathcal{O})$. The generators of the stabilizers will be found in terms of the elements τ, and the υ_i and then Proposition 23 can be used to find the generators in terms of the generating set $\{\alpha_i, \beta_i, \gamma_j\}$ derived from the marking $\{\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_j\}$ on T. Our result is Theorem 27 which splits into two different cases depending on whether $B \in \partial^d(\overline{T})$ is a segment or a circle.

Theorem 27 *Let S be a surface with G-action, Λ be the covering Fuchsian group of G, and $T = S/G$. Let $\mathcal{O}$ be an oval of a symmetry θ on S, which induces a symmetry ψ on T, let B be the part of the dissected boundary $\partial^d(\overline{T})$ to which $\mathcal{O}$ projects, let $\bar{C}$ be the component of the boundary of $\overline{T}$ containing the image $\bar{B}$ of B, and let $\tau, \upsilon_1, \ldots, \upsilon_s \in \Lambda$, and t, $u_1, \ldots, u_s \in G$, be the elements defined above with respect to $\bar{C}$. Then the stabilizers* $\mathrm{Stab}_\Lambda(L)$ *and* $\mathrm{Stab}_G(\mathcal{O})$ *are as follows:*

i) B is a circle: Let the orders of the periodic points be $2\mu_1 + 1, \ldots, 2\mu_s + 1$ as we traverse B in the positive direction. Then $\mathrm{Stab}_\Lambda(L)$ *and* $\mathrm{Stab}_G(\mathcal{O})$ *are cyclic groups conjugate to the following subgroups.*

$$\mathrm{Stab}_\Lambda(L) \simeq \langle \upsilon_1^{\mu_1} \upsilon_2^{\mu_2} \cdots \upsilon_s^{\mu_s} \tau \rangle,$$

and

$$\mathrm{Stab}_G(\mathcal{O}) \simeq \langle u_1^{\mu_1} u_2^{\mu_2} \cdots u_s^{\mu_s} t \rangle$$

ii) B is a segment: Assume that the periodic points $z_k, \ldots, z_{k+b+1}$ along the segment have orders $2\mu_k, 2\mu_{k+1} + 1, \ldots, 2\mu_{k+b} + 1$, $2\mu_{k+b+1}$, where the segment has b odd periodic points in its interior and the numbering of the points on the segment starts at k. Then $\mathrm{Stab}_\Lambda(L)$ *and* $\mathrm{Stab}_G(\mathcal{O})$*are dihedral groups conjugate to the following subgroups.*

$$\mathrm{Stab}_\Lambda(L) \simeq \langle \upsilon_k^{\mu_k}, \omega \upsilon_{k+b+1}^{\mu_{k+b+1}} \omega^{-1} \rangle,$$

and

$$\mathrm{Stab}_G(\mathcal{O}) \simeq \langle u_k^{\mu_k}, w u_{k+b+1}^{\mu_{k+b+1}} w^{-1} \rangle.$$

where $\omega = \upsilon_{k+1}^{\mu_{k+1}} \upsilon_{k+2}^{\mu_{k+2}} \cdots \upsilon_{k+b}^{\mu_{k+b}}$ *and* $w = u_{k+1}^{\mu_{k+1}} u_{k+2}^{\mu_{k+2}} \cdots u_{k+b}^{\mu_{k+b}}$.

We will use Theorem 27 through the following Corollary.

Corollary 28 *Adopt the notation of Theorem 27. Then the orders of stabilizers of ovals are given as follows:*

i) B is a circle: Let M be the order of the element $h = u_1^{\mu_1} u_2^{\mu_2} \cdots u_s^{\mu_s} t$ in G. Then h generates a conjugate of $\mathrm{Stab}_G(\mathcal{O})$ *and*

$$|\mathrm{Stab}_G(\mathcal{O})| = M.$$

ii) B is a segment: Let $w = u_{k+1}^{\mu_{k+1}} u_{k+2}^{\mu_{k+2}} \cdots u_{k+b}^{\mu_{k+b}}$, and let M be the order of the element $h = u_k^{\mu_k} w u_{k+b+1}^{\mu_{k+b+1}} w^{-1}$. Then h generates the rotational subgroup of a conjugate of $\mathrm{Stab}_G(\mathcal{O})$ *and*

$$|\mathrm{Stab}_G(\mathcal{O})| \simeq 2M.$$

Proof of Theorem 27 *The circle case:* The idea of the proof is as follows (see Figure 5 following the proof). Pick an edge e_0 of the master tile projecting to $\overline{B}$ and let L be the line determined by e_0. This line projects to a conjugate of $\mathcal{O}$. Using a sequence of reflection products, we "roll" the master tile along the line L until the edge $e_s = \tau e_0$ rests on L. Let ω be the product of reflections such that ωe_s lies on L. Then, both e_0 and $\omega\tau e_0$ lie on L, hence $\omega\tau$ is a translation of L. As we shall see in our construction, all the edges on L from e_0 up to but not including $\omega\tau e_0$ are Λ-inequivalent, from which we may conclude $\text{Stab}_\Lambda(L) = \langle\omega\tau\rangle$. It remains to compute ω by examining the details of the "rolling" the tile. The process is illustrated for $s = 2$ in Figure 5, where the initial tile and the subsequent tiles obtained by rolling are drawn. Assume that the orientation conditions given in (5.4) and (5.5) hold, and let y_i, e_i, y and z be as in §5. If C has no periodic points then obviously $\tau = \xi_\Lambda(\epsilon)$ generates the translation group of L since $L/\langle\tau\rangle = \bar{C}$. The period at the point y_j, may be written $2\mu_j + 1$, since all periods are odd. The reflection product $r_1 r_0$ is a clockwise rotation through $2\pi/(2\mu_1 + 1)$ about the point y_1 so that $\omega_1 = (r_1 r_0)^{\mu_1}$ rotates or rolls U_0 onto the tile $U_1 = \omega_1 U_0$ whose intersection with L is $\omega_1 e_1$. The two edges of U_1 that meet at $\omega_1 y_2$ are $\omega_1 e_1$ and $\omega_1 e_2$. The corresponding reflections at these edges are $\omega_1 r_1 \omega_1^{-1}$ and $\omega_1 r_2 \omega_1^{-1}$ and the corresponding clockwise rotation through $2\pi/(2\mu_2 + 1)$, centred at $\omega_1 y_2$ is $\omega_1 r_2 r_1 \omega_1^{-1} = \omega_1 r_2 \omega_1^{-1} \omega_1 r_1 \omega_1^{-1}$. Set $\omega_2 = (r_2 r_1)^{\mu_2}$. The rotation $\omega_1 \omega_2 \omega_1^{-1} = \omega_1 (r_2 r_1)^{\mu_2} \omega_1^{-1} = (\omega_1 r_2 r_1 \omega_1^{-1})^{\mu_2}$ carries U_1 to the tile $U_2 = \omega_1 \omega_2 \omega_1^{-1} U_1 = \omega_1 \omega_2 \omega_1^{-1} \omega_1 U_0 = \omega_1 \omega_2 U_0$. Repeat the procedure to get a sequence of tiles $\{U_i\}$ such that U_i meets L in the segment $\omega_1 \cdots \omega_i e_i$, where $\omega_i = (r_i r_{i-1})^{\mu_i}$. It follows that U_s meets L in the edge $\omega_1 \cdots \omega_s e_s = \omega_1 \cdots \omega_s \tau e_0$. From the construction $\omega\tau = \omega_1 \cdots \omega_s \tau$ is a translation of L and $L/\langle\omega\tau\rangle$ is the image of the union of the segments e_0, $\omega_1 e_1$, $\omega_1\omega_2 e_2$, $\ldots$, $\omega_1 \cdots \omega_{s-1} e_{s-1}$. These segments are all Λ-inequivalent so that $\omega\tau$ generates the group of translations of L lying in Λ. Putting the definitions together $\omega\tau = (r_1 r_0)^{\mu_1} (r_2 r_1)^{\mu_2} \cdots (r_s r_{s-1})^{\mu_s} \tau = \upsilon_1^{\mu_1} \upsilon_2^{\mu_2} \cdots \upsilon_s^{\mu_s} \tau$. This proves the statement about $\text{Stab}_\Lambda(L)$, the statement about $\text{Stab}_G(\mathcal{O})$ follows from applying the epimorphism $\eta : \text{Stab}_\Lambda(L) \to \text{Stab}_G(\mathcal{O})$.

The segment case: Suppose that B is a segment and that the edges have been enumerated as above. Let L be the line determined by e_k, up to conjugacy we may assume that L projects to $\mathcal{O}$. Since B is a segment then $\text{Stab}_\Lambda(L)$ is an infinite dihedral group. This group is generated by any two involutions ι_1, $\iota_2 \in \Lambda$ such that there is no other involution $\iota_3 \in \Lambda$ of L whose fixed point lies in between the fixed points of ι_1, ι_2. For ι_1 we choose $\omega_k = (r_k r_{k-1})^{\mu_k}$ which is clearly an involution since the angle between e_{k-1} and e_k is $\pi/(2\mu_k)$. Now follow the construction in the circle case with b stages for the odd order points

$y_{k+1}, \ldots y_{k+b}$. The transformation $\omega = \omega_{k+1} \cdots \omega_{k+b}$ maps e_{k+b} to a segment on the line L. Thus the point $\omega_{k+1} \cdots \omega_{k+b} y_{k+b+1}$ lies on L since y_{k+b+1} is an endpoint of e_{k+b}. Set $\omega_{k+b+1} = (r_{k+b+1} r_{k+b})^{\mu_{k+b+1}}$ and $\omega = \omega_{k+1} \cdots \omega_{k+b}$. Then ω_{k+b+1} is an involution since y_{k+b+1} is an even periodic point of order $2\mu_{k+b+1}$, and $\iota_2 = \omega\omega_{k+b+1}\omega^{-1}$ is an involution with fixed point ωy_{k+b+1} lying on L. By our construction all the periodic points on L between y_k and ωy_{k+b+1} have odd order so that $\mathrm{Stab}_\Lambda(L) = \langle \iota_1, \iota_2 \rangle$ and this case is finished. ■

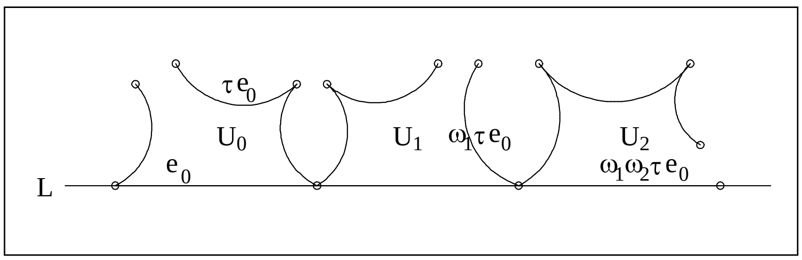

Fig 5. Rolling the tile

Genus 3 oval calculations for $G = \mathbb{Z}_3$ In order to determine the number of ovals we first have to determine what terms in (3.1) are relevant and then compute the stabilizer quotients in (3.2), using Corollary 28. From the earlier construction of our $\overline{T}$ there is only one boundary component with all periods odd. Hence there is one term in (3.1). We note that action of the symmetry Θ on G is the inversion automorphism and hence the centralizer of θ is trivial. Therefore, the generator of the rotation group should be trivial. For Case 5-0

$$wt = c_1^{-1} c_2^{-1} c_3^{-1} c_4^{-1} c_5^{-1} = 1,$$

as predicted.

For cases Cases 3-2 and 1-4 we get

$$wt = c_1^{-1} c_2^{-1} c_3^{-1} c_4 = 1,$$

and

$$wt = c_1^{-1} c_2 c_3 = 1$$

respectively for the appropriate vectors listed in (4.18). Thus there is one oval in every case.

7 Symmetric genus 3 surfaces with Σ_4-action

In this section we describe the family of symmetric genus 3 surfaces whose automorphism group is Σ_4 and which have at least one symmetry with an oval (normalizing Σ_4). Our calculation will include the determination of the conjugacy classes of symmetries with ovals, and the number of ovals. To make the discussion reasonably complete we will also describe the *moduli space* of conformal types of surfaces with Σ_4-action and the subset of symmetric surfaces.

The Σ_4 actions on surfaces of genus 3 From [2] we may classify all of surfaces of genus 3 with Σ_4-action as in Proposition 29 below. Two conformal G-actions $\varepsilon : G \to \mathrm{Aut}(S)$, and $\varepsilon' : G \to \mathrm{Aut}(S')$, are conformally equivalent if and only if there is a conformal isomorphism $h : S \to S'$ and a group automorphism $\omega \in \mathrm{Aut}(G)$ such that $\epsilon'(g) = h \circ \varepsilon(\omega(g)) \circ h^{-1}$ for all $g \in G$. If $S = S'$ the actions are conformally equivalent if and only if $\varepsilon(G)$ and $\varepsilon'(G)$ are conjugate in $\mathrm{Aut}(S)$.

Proposition 29 *If S is a genus 3 surface with Σ_4-action then exactly one of the following holds:*

i) The full automorphism group of S is Σ_4, and the branching data is $(2,2,2,3)$.

ii) The full automorphism group of S is $\mathbb{Z}_2 \times \Sigma_4$ with branching data $(2,4,6)$. *The surface S admits a* $(3,4,4)$*-action and a* $(2,2,2,3)$*-action of Σ_4 both unique up to conformal equivalence.*

iii) The full automorphism group of S is $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ with branching data $(2,3,8)$. *The surface S admits a* $(2,2,2,3)$*-action of Σ_4, unique up to conformal equivalence, but no* $(3,4,4)$*-actions*

iv) The full automorphism group of S is $PSL_2(7)$ with branching data $(2,3,7)$. *The surface S admits two conformal equivalence classes of* $(2,2,2,3)$*-actions of Σ_4, but no* $(3,4,4)$*-actions.*

Proof. According to [2] there are two types Σ_4-actions on surfaces of genus 3, their branching data are $(3,4,4)$ and $(2,2,2,3)$. The surface with $(3,4,4)$-action is unique up to conformal equivalence, and as we shall see, the automorphism group of this surface strictly contains Σ_4. This proves *i*). Now Suppose that $\mathrm{Aut}(S)$ strictly contains Σ_4. From [2] the automorphism groups of genus 3 surfaces, strictly divisible by 24, are the groups $\mathbb{Z}_2 \times \Sigma_4$, $\mathbb{Z}_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$, $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ and $PSL_2(7)$. We may eliminate $\mathbb{Z}_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ since Σ_4 must then either lie in $\mathbb{Z}_4 \times \mathbb{Z}_4$ or have $\mathbb{Z}_3$ as a quotient, both of which are impossible. We consider the remaining cases one at a time.

Let's examine $\mathbb{Z}_2 \times \Sigma_4$ first. This group contains two normal copies of Σ_4, the obvious one and the subgroup $\Sigma_4' = \{1\} \times A_4 \cup \{-1\} \times \tau A_4$ where we write $\mathbb{Z}_2 = \{1, -1\}$, and τ is a transposition. As calculated in [2], each generating vector of $\mathrm{Aut}(S)$ has the form $(t\tau, y, t\sigma)$ where t generates $\mathbb{Z}_2$ and y and σ are cycles of order 4 and 3 in Σ_4, respectively. An element of $\mathbb{Z}_2 \times \Sigma_4$ will have a fixed point of order $2, 4$, or 6 if and only if it is conjugate to $t\tau$, y or $t\sigma$, respectively and hence elements conjugate to y^2 and $\sigma^2 = (t\sigma)^2$ have fixed points of order 2 and 3 respectively. By considering the types of elements in the two subgroups we see that Σ_4 has elements with fixed points of order 4 and that Σ_4' has no elements with fixed points of order 4. It follows that the branching data of Σ_4 and Σ_4' must be $(3, 3, 4)$ and $(2, 2, 2, 3)$, respectively. The actions are unique since there is only one subgroup of each type.

Now suppose $\mathrm{Aut}(S) = \Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$. By computation, using the GAP software [8], $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ has 4 conjugate subgroups isomorphic to Σ_4, a representative of which is $\Sigma_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Each of the four conjugate Σ_4 must have the same branching data, and the actions are conformally equivalent. Since Σ_4 is self-normalizing in $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$, the latter group does not contain a copy of $\mathbb{Z}_2 \times \Sigma_4$. Since the surface with the $(3, 3, 4)$-action of Σ_4 is unique up to conformal equivalence and has automorphism group $\mathbb{Z}_2 \times \Sigma_4$, it follows that the branching data Σ_4 must be $(2, 2, 2, 3)$.

Finally suppose $\mathrm{Aut}(S) = PSL_2(7)$. The group $PSL_2(7)$ has two conjugacy classes of Σ_4's which are interchanged by every non-trivial outer automorphism of $PSL_2(7)$. Each class defines a conformal equivalence class of actions on S. Since the two classes are not conjugate in $PSL_2(7)$ and $PSL_2(7)$ is the full automorphism group. ∎

Remark 30 *The surfaces in i) form a complex one parameter family of conformally inequivalent surfaces and the surfaces in ii), iii) and iv) are unique up to conformal equivalence.*

Moduli of Σ_4 actions We need to find a model for the family of conformal equivalence classes of surfaces that satisfy *i)* of the above proposition and then describe the subset of it that corresponds to symmetric surfaces. We do this in three stages. First we find a model $\mathcal{M}_T$ for the possible quotients $T = S/G$ and then produce the family of all surfaces with Σ_4-$(3,2,2,2)$ action as a branched cover $q : \mathcal{M}_G \to \mathcal{M}_T$. From this cover we remove a finite number of degenerate fibres arising from the surfaces of higher symmetry corresponding to *ii*), *iii*) and *iv*) of Proposition 29. Unfortunately, the complete discussion of the symmetric surfaces in these degenerate fibres, though interesting, would take us beyond the intended scope of this paper. Finally we identify the subset of "generic" symmetric surfaces as a unbranched cover $q^s : (\mathcal{M}_G^s)^\circ \to (\mathcal{M}_T^s)^\circ$ of an easily identified subset of $(\mathcal{M}_T^s)^\circ$ of $\mathcal{M}_T$. The fibres of q^s will be proper subsets of the corresponding fibres of q.

The quotient space $T = S/G$ is a sphere with four branch points Q_1, Q_2, Q_3, Q_4 which for convenience we assume have orders $3,2,2,2$, respectively. We take the compactified complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, to be our model of the sphere T. Let $s,p,q,r \in \widehat{\mathbb{C}}$ represent the points Q_1, Q_2, Q_3, Q_4, respectively. There is a unique fractional linear transformation:

$$\varphi : z \to \frac{(q-2r+p)z + (qr - 2qp + pr)}{(p-q)(z-r)}$$

such that $\varphi(p) = 1$, $\varphi(q) = -1$, $\varphi(r) = \infty$. Define the quantity λ by:

$$\lambda = \varphi(s) = \frac{sq - 2sr + sp + qr - 2qp + pr}{(p-q)(s-r)}.$$

Observe that $\lambda \in \widehat{\mathbb{C}} - \{1,-1,\infty\}$. The quotient T is conformally equivalent to the sphere marked at $\{\lambda,-1,1,\infty\}$, which we denote by T_λ. If we relabel the branch points of order 2 then another value of λ, say λ', is determined, though the corresponding quotient $T_{\lambda'}$ is conformally equivalent to T_λ. The set of possible λ's is $\{\lambda, -\lambda, \frac{\lambda+3}{\lambda-1}, -\frac{\lambda+3}{\lambda-1}, \frac{\lambda-3}{\lambda+1}, -\frac{\lambda-3}{\lambda+1}\}$, i.e., an orbit of λ under the obvious finite group F – isomorphic to Σ_3 – of linear fractional transformations of $\widehat{\mathbb{C}}$. We may construct a model for $\mathcal{M}_T$ as the quotient $(\widehat{\mathbb{C}} - \{1,-1,\infty\})/F$, by using the orbit product $\nu = -\frac{\lambda^2(\lambda^2-9)^2}{(\lambda^2-1)^2}$ as a parameter. The quotient map is:

$$m : \widehat{\mathbb{C}} - \{1,-1,\infty\} \to \mathbb{C} = (\widehat{\mathbb{C}} - \{1,-1,\infty\})/F \qquad (7.1)$$

defined by

$$m(\lambda) = \nu = -\frac{\lambda^2(\lambda^2-9)^2}{(\lambda^2-1)^2}.$$

This map is branched over $\{0, 27\}$ corresponding to the non-regular F-orbits $\{3, 0, -3\}$ and $\{\sqrt{3}i, -\sqrt{3}i\}$.

Now we construct $\mathcal{M}_G$ as a branched cover of $\mathcal{M}_T$. Given $\nu \in \mathcal{M}_T$ there exists a Λ_λ, unique up to conjugacy in $\mathrm{Aut}(\mathbb{H})$, such that $\mathbb{H}/\Lambda_\lambda = T_\lambda$. As shown in the generating vector calculation below, there are 9 Aut(G)-inequivalent epimorphisms of Λ_λ onto G. Thus nine surface groups $\Gamma_1, \ldots, \Gamma_9$ are determined as kernels which in turn yields nine surfaces $S_1 = \mathbb{H}/\Gamma_1, \ldots, S_9 = \mathbb{H}/\Gamma_9$. This gives us the nine potential points of the fibre $q^{-1}(\nu) \subset \mathcal{M}_G$ lying above the point $\nu \in \mathcal{M}_T$. A finite number of the fibres above will contain surfaces that are conformally equivalent to each other (as Σ_4-spaces) and these points need to be identified in $\mathcal{M}_G$. Instead of dealing with the tricky points of determining exactly which surfaces are conformally equivalent and carefully defining the topological structure on these special fibres let us content ourselves determining the largest subset of $\mathcal{M}_T^\circ$ of $\mathcal{M}_T$ over which the projection $\mathcal{M}_G \to \mathcal{M}_T$ is unramified as a set mapping. Over $\mathcal{M}_T^\circ$ one may check that there is a monodromy action of $\pi_1(\mathcal{M}_T^\circ, \nu)$ on the fibres of $\mathcal{M}_G \to \mathcal{M}_T$ and hence that $\mathcal{M}_G^\circ = q^{-1}(\mathcal{M}_T^\circ)$ may be topologized as a covering space of $\mathcal{M}_T^\circ$, and hence is a smooth algebraic curve with finitely many punctures.

The following proposition, whose proof we defer to the end of the section, identifies the special values of ν.

Proposition 31 *For each $\nu \in \mathbb{C}$ let $\mathcal{M}_\nu$ denote the set of conformal equivalence classes of genus 3 surfaces with $(3,2,2,2)$-Σ_4 action which determine ν as above. If $\nu \notin \{0,\ 27,\ \frac{46225}{1458},\ \frac{-6656186915}{23887872} + \frac{1076146307\sqrt{7}}{7962624},\ \frac{-6656186915}{23887872} - \frac{1076146307\sqrt{7}}{7962624}\}$ then $\mathcal{M}_\nu$ consists of 9 conformally inequivalent surfaces all of whose automorphism group is exactly Σ_4.*

Determination of the symmetric surfaces Now we consider symmetric surfaces. If S has a symmetry ϕ with fixed points then the quotient symmetry ψ is inversion in a circle and $\overline{T}$ must be a disc. There are two types of symmetric quotient surfaces $\overline{T}$ for the given branching data.

Case 4-0	four symmetric points, in the cyclic order, $(3,2,2,2)$
Case 2-2	two symmetric points and one involutary pair. $(3,2,2\&2)$

In particular, T will admit symmetries only for certain values of λ, since ψ must fix λ and permute $\{-1,1,\infty\}$. The possible symmetries, fixed circles and the admissable λ on the fixed circles are given in the table below. It is easily calculated that T admits two symmetries when $\lambda \in \{3,0,-3\}$, three symmetries when $\lambda \in \{\sqrt{3}i, -\sqrt{3}i\}$, and at most one symmetry otherwise.

Case	$\psi(z)$	fixed circle	Symmetric Points	Involutary Pairs	admissable λ on fixed circle
$\psi_{4,0}$	$\bar{z}$	$\mathbb{R} \cup \{\infty\}$	$1, -1, \infty, \lambda$		$\lambda \notin \{1, -1\}$
$\psi_{2,2,a}$	$-\bar{z}$	$i\mathbb{R} \cup \{\infty\}$	∞, λ	$\{1, -1\}$	all λ
$\psi_{2,2,b}$	$\frac{\bar{z}+3}{\bar{z}-1}$	$(x-1)^2 + y^2 = 4$	$-1, \lambda$	$\{1, \infty\}$	$\lambda \neq -1$
$\psi_{2,2,c}$	$\frac{-\bar{z}+3}{\bar{z}+1}$	$(x+1)^2 + y^2 = 4$	$1, \lambda$	$\{-1, \infty\}$	$\lambda \neq 1$

Using the group F introduced above we can restrict values of λ and the symmetries that we need to consider. The transformations of F map the fixed circle of $\psi_{4,0}$ to itself and $\{\lambda : 0 \leq \lambda < 1\}$ is a fundamental domain for the action. The group also maps the fixed circles of $\psi_{2,2,a}$, $\psi_{2,2,b}$ and $\psi_{2,2,c}$ to each other, a fundamental domain is $\{\lambda i : 0 \leq \lambda < \infty\}$. Thus we need only discuss the symmetries $\psi_{4,0}$ and $\psi_{2,2,a}$ when we compute ovals later. The quotient map m defined in 7.1 above maps the admissable λ for $\psi_{4,0}$ to the interval $(-\infty, 0]$ and the admissable λ for $\psi_{2,2,a}$, $\psi_{2,2,b}$ and $\psi_{2,2,c}$ to $[0, +\infty)$. Thus $\mathcal{M}_T^s = \mathbb{R}$, and according to Proposition 31 we should choose $(\mathcal{M}_T^s)^\circ = \mathbb{R} - \{0, 27, \frac{46225}{1458}\}$. The corresponding subset $(\mathcal{M}_G^s)^\circ$ of symmetric surfaces cannot be defined until we carry out our symmetry analysis below.

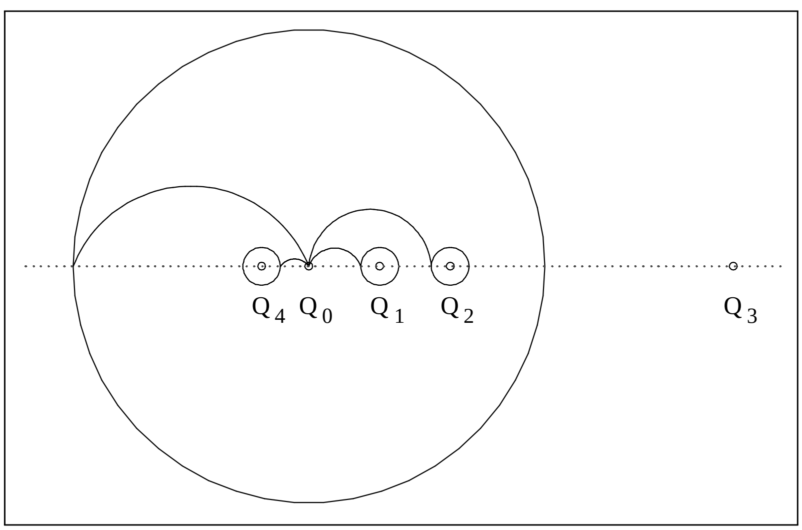

Figure 7.a.: The reflection $\psi_4, 0$

To work with the symmetric surfaces we shall select a single marking for all our computations since we will want to compare the two symmetries $\psi_{4,0}$ and $\psi_{2,2,a}$

for the same set of generating vectors. We pick for Q_0 a point on the real axis lying between -1 and 0. We also restrict λ to the fundamental region on the fixed circles as above. The loops of the marking $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4\}$ are as given in Figures 7.a, 7.b below, following the procedure for selecting a marking given in §4. Now Q_0 is a fixed point of $\psi_{4,0}$, so we need only pick a curve δ for the case $\psi_{2,2,a}$. Choose δ to be a curve from Q_0 to a point R_0 on the negative imaginary axis followed by the mirror image of this curve from R_0 to $\psi_{2,2,a}(Q_0)$. For the case $\psi_{4,0}$ we model the disc $\overline{T}$ as the compactified upper half plane and then we encounter the branch points in the order $\lambda, 1, \infty, -1$ as we travel around the boundary of the circle. In the second case take the compactified left half plane as our model of $\overline{T}$, with branch points λ and ∞ on the boundary and -1 in the interior.

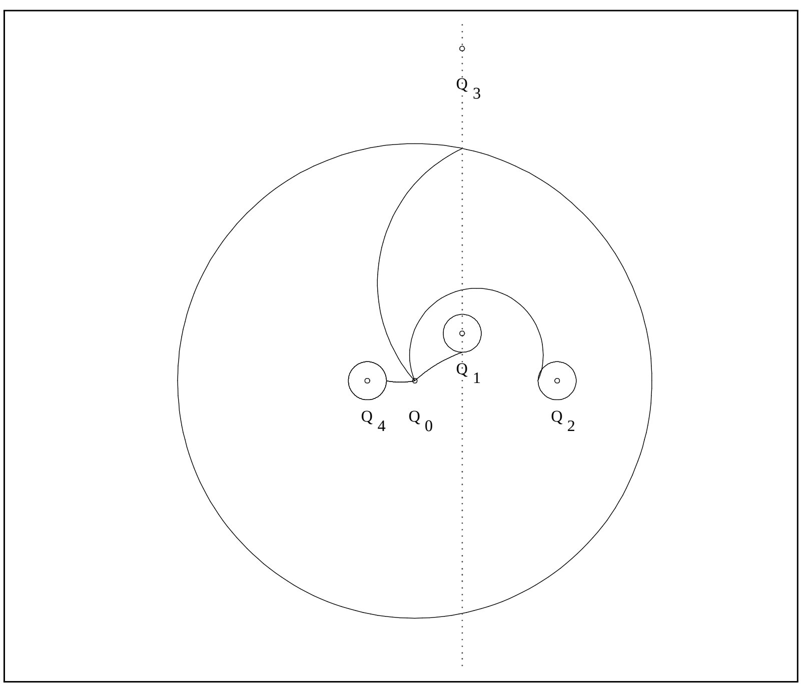

Figure 7.b. The reflection $\psi_2, 2$

The action of Φ on the generating vectors is given in the following table:

Symmetry Action

$\psi_{4,0}$	$\psi_{2,2,a}$
$c_1 \to c_1^{-1}$	$c_1 \to c_1^{-1}$
$c_2 \to c_1 c_2^{-1} c_1^{-1}$	$c_2 \to c_1 c_4^{-1} c_1^{-1}$
$c_3 \to c_4^{-1} c_3^{-1} c_4$	$c_3 \to c_1 c_3^{-1} c_1^{-1}$
$c_4 \to c_4^{-1}$	$c_4 \to c_1 c_2^{-1} c_1^{-1}$

For $\psi_{4,0}$ there are three symmetries to consider, $\theta_0, \theta_1, \theta_2$, whereas for $\psi_{2,2,a}$ we need only look at θ_0. As noted in §4 these may be calculated in terms of a general generating vector, using the formulas in Proposition 14. The elements t and u_i may also be calculated, as well as generators of the rotational subgroup of the oval centralizers: The needed results are summarized in the table below:

Various Group Elements

	$\psi_{4,0}$		$\psi_{2,2,a}$
d_1	$c_1 c_2$		
d_2	$c_1 c_2 c_3$		
t	1	t	c_4
u_0	c_4^{-1}	u_0	$c_4^{-1} c_3^{-1} c_4$
u_1	c_1^{-1}	u_1	c_1^{-1}
u_2	c_2^{-1}	u_2	c_3^{-1}
u_3	c_3^{-1}	u_3	$c_4 c_1^{-1} c_4^{-1}$
$u_4 = u_0$	c_4^{-1}	u_4	$c_4 c_3^{-1} c_4^{-1}$
$u_5 = u_1$	c_1^{-1}	u_5	$c_4^2 c_1^{-1} c_4^{-2}$
$u_6 = u_2$	c_2^{-1}	u_6	$c_4^2 c_3^{-1} c_4^{-2}$

Generators of Rotational Oval Stabilizers

Symmetry	$\psi_{4,0}$	$\psi_{2,2,a}$
θ_0	$c_4^{-1} c_1^{-1} c_2^{-1} c_1$	$c_4^{-1} c_3^{-1} c_4 c_1^{-1} c_3^{-1} c_1$
θ_1	$c_2^{-1} c_3^{-1}$	
θ_2	$c_3^{-1} c_4^{-1}$	

Though the calculation is a fairly straightforward application of previous results, we give sample calculations of the oval stabilizers for θ_0 to illustrate the numbering system. As we move around the oval of ψ in the positive direction the branch points are cyclically labelled $y_1 = \lambda$, $y_2 = 1$, $y_3 = \infty$, $y_4 = -1$, $y_5 = \lambda$, $y_6 = 1$. To apply the segment case to the oval determined by the portion from -1 to 1, we describe it as $\{y_4, y_5, y_6\}$, so that $\mu_4 = \mu_5 = \mu_6 = 1$, w

$= u_5$, and the stabilizer is $u_4 w u_6 w^{-1} = u_4 u_5 u_6 u_5^{-1}$as required. The numbering for $\psi_{2,2,a}$ is different: $y_1 = \lambda$, $y_2 = \infty$, $y_3 = \lambda$, $y_4 = \infty$, $y_5 = \lambda$, $y_6 = \infty$. A similar sample calculation yields $h = u_0 u_1 u_2 u_1^{-1} = c_4^{-1} c_3^{-1} c_4 c_1^{-1} c_3^{-1} c_1$

We are now ready to do our computations with generating vectors. As this involves significant amount of computation, the calculations were done using GAP [8] and MAGMA [9] software. There are 9 $\mathrm{Aut}(G)$ classes of generating vectors of Σ_4; representatives of the classes are as given in Table 7.1.

Table 7.1

$(3,2,2,2)$ **Generating Vectors for** Σ_4

$\mathbf{v}_1 = ((2,3,4),(1,2)(3,4),(2,4),(1,4))$
$\mathbf{v}_2 = ((2,3,4),(1,2)(3,4),(1,2),(2,4))$
$\mathbf{v}_3 = ((2,3,4),(1,2)(3,4),(1,4),(1,2))$
$\mathbf{v}_4 = ((2,3,4),(3,4),(1,3)(2,4),(1,3))$
$\mathbf{v}_5 = ((2,3,4),(1,2),(1,2)(3,4),(2,4))$
$\mathbf{v}_6 = ((2,3,4),(1,2),(1,4)(2,3),(1,3))$
$\mathbf{v}_7 = ((2,3,4),(3,4),(1,3),(1,3)(2,4))$
$\mathbf{v}_8 = ((2,3,4),(1,2),(2,4),(1,4)(2,3))$
$\mathbf{v}_9 = ((2,3,4),(1,2),(1,3),(1,2)(3,4))$

Since all automorphisms of Σ_4 are inner then we may rewrite G^* as $\mathbb{Z}_2 \times G$ where $\mathbb{Z}_2 = \langle s \rangle$ and $\theta_0 = s\iota$ for some element ι inducing Θ_0 by conjugation. Then θ_i is given by $s\iota d_i$. Since s is central all involutions of G^* that do not lie in G have the form se where $e^2 = 1$ in G, and hence is conjugate to exactly one of one of $(\ )$, $(1,2)$ or $(1,2)(3,4)$ with the following orders of centralizers:

Orders of Centralizers of se

e	$(\ )$	$(1,2)$	$(1,2)(3,4)$
$\lvert\mathrm{Cent}_G(se)\rvert$	24	4	8

Now using the criterion of Proposition 14 we can determine which generating vectors yield symmetries and the conjugacy class representative for the various θ_i. The results for $\psi_{4,0}$ are given in the tables following. In Table 7.2.a if two of $\theta_0, \theta_1, \theta_2$, are conjugate we use the symbol $\backsim$ to denote the conjugacy relationship. In Table 7.2.b we have recorded the computed generator h_i of the rotational subgroup of the stabilizer of an oval.

Table 7.2.a
Symmetry Classes for $\psi_{4,0}$

	θ_0	θ_1	θ_2
$\mathbf{v}_1$			
$\mathbf{v}_2$	$(2,4)s \backsim \theta_1$	$(1,2)s \backsim \theta_0$	$(\,)s$
$\mathbf{v}_3$			
$\mathbf{v}_4$	$(2,4)s$	$(\,)s$	$(1,3)(2,4)s$
$\mathbf{v}_5$	$(2,4)s$	$(1,2)(3,4)s$	$(\,)s$
$\mathbf{v}_6$	$(2,4)s$	$(1,2)(3,4)s \backsim \theta_2$	$(1,3)(2,4)s \backsim \theta_1$
$\mathbf{v}_7$	$(2,4)s \backsim \theta_2$	$(\,)s$	$(1,3)s \backsim \theta_0$
$\mathbf{v}_8$			
$\mathbf{v}_9$			

Table 7.2.b
Generators of Rotational Oval Stabilizers

	h_0	h_1	h_2
$\mathbf{v}_1$			
$\mathbf{v}_2$	$(1,3)$	$(3,4)$	$(1,4,2)$
$\mathbf{v}_3$			
$\mathbf{v}_4$	$(1,3)(2,4)$	$(1,3,2,4)$	$(2,4)$
$\mathbf{v}_5$	$(1,3)(2,4)$	$(3,4)$	$(1,4,3,2)$
$\mathbf{v}_6$	$(\,)$	$(1,3,2,4)$	$(1,4,3,2)$
$\mathbf{v}_7$	$(1,3)$	$(1,3,4)$	$(2,4)$
$\mathbf{v}_8$			
$\mathbf{v}_9$			

The corresponding data for $\psi_{2,2,a}$ is simpler and given in Table 7.3 below.

From the data in the tables 7.1, 7.2 and 7.3 we easily calculate the numbers of ovals. As noted above two of $\theta_0, \theta_1, \theta_2$ are conjugate in several of the cases above. For those entries the number of ovals has to be computed by taking the sum of oval contributions $\frac{|\mathrm{Cent}_G(\theta_i)|}{|\mathrm{Stab}_G(\mathcal{O}_i)|}$ over the pairs of conjugate reflections. The results are given in Table 7.4.

Remark 32 *Even though T has three symmetries for $\lambda \in \{\sqrt{3}i, -\sqrt{3}i\}$ no generating vector is preserved by more than one symmetry. To see this observe that each of the symmetries moves c_i to a conjugate of c_j, where the symmetry moves Q_i to Q_j. In each generating vector one of the elements of order 2 is* *not conjugate to the other two. Thus the symmetry must fix the corresponding point. However none of the three symmetries fix the same point.*

Table 7.3
Symmetry Classes for $\psi_{2,2,a}$ and
Generators of Rotational Oval Stabilizers

	θ_0	h_0
$\mathbf{v}_1$		
$\mathbf{v}_2$		
$\mathbf{v}_3$		
$\mathbf{v}_4$		
$\mathbf{v}_5$		
$\mathbf{v}_6$	$(2,3)s$	$(\,)$
$\mathbf{v}_7$		
$\mathbf{v}_8$		
$\mathbf{v}_9$		

Table 7.4
Numbers of Ovals for Symmetry Classes

$\psi_{4,0}$	θ_0	θ_1	θ_2	$\psi_{2,2,a}$	θ_0
$\mathbf{v}_1$				$\mathbf{v}_1$	
$\mathbf{v}_2$	2	2	4	$\mathbf{v}_2$	
$\mathbf{v}_3$				$\mathbf{v}_3$	
$\mathbf{v}_4$	1	3	2	$\mathbf{v}_4$	
$\mathbf{v}_5$	1	2	3	$\mathbf{v}_5$	
$\mathbf{v}_6$	2	2	2	$\mathbf{v}_6$	2
$\mathbf{v}_7$	2	4	2	$\mathbf{v}_7$	
$\mathbf{v}_8$				$\mathbf{v}_8$	
$\mathbf{v}_9$				$\mathbf{v}_9$	

Let us summarize the information in the tables above into a couple of theorems.

Theorem 33 *Let S be a generic genus 3 surface with $G = \Sigma_4$ as automorphism group such that S/G is a sphere with branch points at $\lambda, 1, -1, \infty$, and let $\nu = -\frac{\lambda^2(\lambda^2-9)^2}{(\lambda^2-1)^2}$. If $\nu \notin \{0,\, 27,\, \frac{46225}{1458},\, \frac{-6656186915}{23887872} + \frac{1076146307\sqrt{7}}{7962624},\, \frac{-6656186915}{23887872} - \frac{1076146307\sqrt{7}}{7962624}\}$ then there are nine conformally inequivalent surfaces whose generating vectors are as given in Table 7.1. Furthermore, if $-\infty < v < 0$ then the symmetric surfaces and conjugacy classes of symmetries are given by Table 7.2,* *and if* $0 < v < \frac{46225}{1458}$ *or* $\frac{46225}{1458} < \nu < \infty$ *the symmetric surfaces and conjugacy classes of symmetries are given by Table 7.3. Finally the numbers of ovals for the various classes of symmetries are given by Table 7.4.*

Our last theorem describes the family of generic symmetric surfaces as a covering space.

Theorem 34 *Let* $\mathcal{M}_G^\circ \to \mathcal{M}_T^\circ$ *be the fibre description of the space of the generic genus 3 surfaces with* Σ_4 *as automorphism group as given in Proposition 31 above. Then the set of generic, symmetric, genus 3 surfaces, with* Σ_4 *as automorphism group, is a covering space* $q^s : (\mathcal{M}_G^s)^\circ \to (\mathcal{M}_T^s)^\circ$ *where* $(\mathcal{M}_T^s)^\circ = \mathbb{R}- \{0, 27, \frac{46225}{1458}\}$. *For each* $\nu \in (\mathcal{M}_T^s)^\circ$ *the number of conformally inequivalent symmetric surfaces in* $(q^s)^{-1}(\nu)$ *is given as follows:*

$$-\infty < v < 0, \ |(q^s)^{-1}(\nu)| = 5,$$
$$0 < v < \frac{46225}{1458}, \frac{46225}{1458} < \nu < \infty, \ |(q^s)^{-1}(\nu)| = 1.$$

The last detail of our paper is the following proof.

Proof of Proposition 31 Suppose that ν is chosen as described and that two of the surfaces in the same fibre are conformally equivalent. Then the corresponding Γ_i's are conjugate in $\mathrm{Aut}(\mathbb{H})$, say $\gamma\Gamma_i\gamma^{-1} = \Gamma_j$. It follows that the normalizers of Γ_i and Γ_j are conjugate in $\mathrm{Aut}(\mathbb{H})$ via γ. We consider two separate cases depending on whether Σ_4 is the full automorphism group of S_i or not. Note that the normalizers of the various Γ_i need not all be equal. Let us first consider the case where Σ_4 is the full automorphism group of S_i. It follows that Λ_λ is the normalizer of both Γ_i and Γ_j, and hence $\gamma\ \Lambda_\lambda\gamma^{-1} = \Lambda_\lambda$. Since no two Γ_i's are conjugate in the Λ_λ, it follows that $\gamma \notin \Lambda_\lambda$ and that γ induces an automorphism of T_λ. This can only happen if there is a conformal automorphism of S^2 fixing λ and permuting $\{1, -1, \infty\}$ which in turn forces $\nu \in \{0, 27\}$.

Next we consider the case where Σ_4 is not the full automorphism group. If $\Sigma_4 \lhd \mathrm{Aut}(\mathbf{S})$ then $\mathrm{Aut}(S)/\ \Sigma_4$ acts nontrivially on T_λ fixing λ and permuting $\{1, -1, \infty\}$ which again forces $\nu \in \{0, 27\}$. This leaves us with the cases where the automorphism group of S_i is $\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ or $PSL_2(7)$. We must show that these cases lead to the remaining values of ν. Let N be the normalizer of Γ_i. Then we have $\Gamma_i \subset \Lambda_\lambda \subset N$. The inclusion relations for the subgroups leads to a factorization $q_1 = fq_2$ of the following maps: $q_1 : \mathbb{H}/\Gamma_i \to \mathbb{H}/N$, $q_2 : \mathbb{H}/\Gamma_i \to \mathbb{H}/\Lambda_\lambda$ and $f : \mathbb{H}/\Lambda_\lambda \to \mathbb{H}/N$. The map f is a rational mapping of S^2 to itself. By analyzing how the branching data of the two spherical quotients $\mathbb{H}/\Lambda_\lambda$ and $\mathbb{H}/N$ are intertwined by f, we may explicitly calculate f and the possible values of λ and ν.

Let us work out the case $\mathrm{Aut}(S) = \Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ in detail, the case for $PSL_2(7)$ is similar Since $q_1 = fq_2$, the ramification points of f must lie above the branch points of q_1, and for each point $P \in S^2$ the product of the branching order of q_2 above P and the branching order of f at P must equal the branching order of q_1 above $f(P)$. Moreover, the fibres of f all have four points when counted with multiplicity, since $[N : \Lambda_\lambda] = [\Sigma_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4) : \Sigma_4] = \frac{96}{24} = 4$. If we assume that the branch points of q_1 are $0, 1$ and ∞ with orders $2, 3$ and 8 respectively, then a little work shows that:

- f has four-fold branching above the point ∞ and that $f^{-1}(\infty)$ is a single 2-fold point of q_2;
- $f^{-1}(1)$ consists of two points one of which is a triply ramified point of f and a simple point for q_2 and the other point is λ, the triple point of q_2 which is not ramified by f; and
- $f^{-1}(0)$ has three points, two of which are double points of q_2 unramified by f and the third point is a simple point for q_2 that is doubly ramified by f.

By precomposing f with an appropriate transformation from F we may assume that:

$$\begin{aligned} f^{-1}(\infty) &= \{\infty, \infty, \infty, \infty\}, \\ f^{-1}(1) &= \{\lambda, b, b, b\}, \\ f^{-1}(0) &= \{a, a, -1, 1\}, \end{aligned}$$

where a point is repeated according to its multiplicity, and a, b are undetermined. Thus f has the form $c(z-a)^2(z^2-1)$. and $cf-1$ factors as $r(z-b)^3(z-\lambda)$ for some r and b. Using a resultant calculation with derivatives (done with MAPLE [10]) we get the two possibilities

$$a = 2i\sqrt{2}, b = \frac{i}{\sqrt{2}}, c = 4/27, \lambda = \frac{5i}{\sqrt{2}}$$

and

$$a = -2i\sqrt{2}, b = -\frac{i}{\sqrt{2}}, c = 4/27, \lambda = -\frac{5i}{\sqrt{2}}$$

In both cases the corresponding value of ν is $\frac{46225}{1458}$.

The analysis for $PSL_2(7)$ similar. The degree of f is $7 = \frac{168}{24}$, and we may assume that the branching orders of $PSL_2(7)$ at $0, 1, \infty$ are $2, 3, 7$ respectively and hence that

$$\begin{aligned} f^{-1}(\infty) &= \{a, a, a, a, a, a, a\} \\ f^{-1}(1) &= \{\lambda, b, b, b, c, c, c\}, \\ f^{-1}(0) &= \{-1, 1, \infty, d, d, e, e\} \end{aligned}$$

for some undetermined values $a, b, c, d, e, f, \lambda$. The rational function then has the form

$$f(z) = r\frac{(z^2-1)(z-d)^2(z-e)^2}{(z-a)^7}$$

such that $f(z) - 1$ satisfies

$$f(z) - 1 = r\frac{(z^2-1)(z-d)^2(z-e)^2 - (z-a)^7}{(z-a)^7} = s\frac{(z-\lambda)(z-b)^3(z-c)^3}{(z-a)^7}$$

for some parameter s. Expanding both sides of the equation and comparing coefficients gives eight algebraic equations in eight unknowns. By appropriately transforming the equations and solving the equations (discussed below, since it is not completely obvious) we get the following possibilities for λ and ν :

$$\begin{aligned} \lambda &= \frac{(245+\sqrt{7}i)\sqrt{294+14\sqrt{7}i}}{3584}, \nu = \frac{-6656186915}{23887872} + \frac{1076146307\sqrt{7}i}{7962624}, \\ \lambda &= \frac{-(245+\sqrt{7}i)\sqrt{294+14\sqrt{7}i}}{3584}, \nu = \frac{-6656186915}{23887872} + \frac{1076146307\sqrt{7}i}{7962624}, \\ \lambda &= \frac{(245-\sqrt{7}i)\sqrt{294-14\sqrt{7}i}}{3584}, \nu = \frac{-6656186915}{23887872} - \frac{1076146307\sqrt{7}i}{7962624}, \\ \lambda &= \frac{-(245-\sqrt{7}i)\sqrt{294-14\sqrt{7}i}}{3584}, \nu = \frac{-6656186915}{23887872} - \frac{1076146307\sqrt{7}i}{7962624}. \end{aligned}$$

The calculation also gives the values of the remaining parameters though we do not give them here.

The system of equations as given were too complex to easily solve, so the following approach was adopted. By precomposing f with a the linear fractional transformation $L(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ satisfying $L(\infty) = a$, $L(0) = b$, and $L(1) = c$,we may assume that $f^* = f \circ L$ satisfies

$$\begin{aligned} f^{*-1}(\infty) &= \{\infty, \infty, \infty, \infty, \infty, \infty, \infty\} \\ f^{*-1}(1) &= \{\lambda^*, 0, 0, 0, 1, 1, 1\}, \\ f^{*-1}(0) &= \{a^*, b^*, c^*, d^*, d^*, e^*, e^*\}, \end{aligned}$$

The transformed function then has the form $f^*(z) = r^*(z - a^*)(z - b^*)(z - c^*)(z - d^*)^2(z - e^*)^2$ where $f^*(z) - 1 = s^*(z - \lambda^*)z^3(z - 1)^3$ for appropriately chosen starred parameters. Now write

$$\begin{aligned} (z - a^*)(z - b^*)(z - c^*) &= z^3 + Az^2 + Bz + C \\ (z - d^*)(z - e^*) &= z^2 + Dz + E \end{aligned}$$

We then have;

$$r^*(z^3 + Az^2 + Bz + C)(z^2 + Dz + E - 1)^2 = f(z) - 1 = s^*(z - \lambda^*)z^3(z - 1)^3.$$

The resulting equations are "more linear" and hence more easily solved. In fact, ordering the variables $s^*, r^*, A, B, C, \lambda^*, D, E$ we may systematically eliminate s^*, r^*, A, B, and C from the equations such that at each stage the variable is replaced by a polynomial in the variables that follow it. The variable λ^* may be eliminated as a rational function in D and E. This leaves two nonlinear polynomials in D and E which Maple [10] can solve. Thus λ^*, A, B, C are known and a^*, b^*, c^* be calculated by solving $z^3 + Az^2 + Bz + C = 0$, using Maple. Since L maps the set $\{a^*, b^*, c^*\}$ to $\{1, -1, \infty\}$, and so L may be determined up to a permutation of $\{1, -1, \infty\}$. This yields six possible values of $\lambda = L(\lambda^*)$, all of which yield the same value of ν. ■

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