Quantification and WFFS of FOL

Quantification

- In English and other natural languages, basic sentences are made by combining noun phrases and verb phrases.
- The simplest noun phrases are names, like Max and Claire, which correspond to the constant symbols of FOL.
- More complex noun phrases are formed by combining common nouns with words known as determiners, such as every, some, most, the, three, and no.
- This gives us noun phrases like *every cube*, *some man from Indiana*, *most children in the class*, *the dodecahedron in the corner*, *three blind mice*, and *no student of logic*.
- Logicians call noun phrases of this sort quantified expressions, and sentences containing them quantified sentences.
- They are so called because they allow us to talk about quantities of things: *every cube*, *most children*, and so forth.

Quantified Sentences

Every rich actor is a good actor.
Brad Pitt is a rich actor.
Brad Pitt is a good actor.

Many rich actors are good actors.
Brad Pitt is a rich actor.
Brad Pitt is a good actor.

No rich actor is a good actor.
Brad Pitt is a rich actor.
Brad Pitt is a good actor.

Hidden Quantification

Max is home whenever Claire is at the library.

- Translate to:
  Every time when Claire is at the library is a time when Max is at home.

Variables

- The language FOL has an infinite number of variables, any of t, u, v, w, x, y, and z, with or without numerical subscripts.
- Variables can occur in atomic wffs in any position normally occupied by a name.
- The program Fitch understands all of these variables; Tarski's World only understands variables u through z without subscripts.

Universal Quantifier (∀)

- In English, we have quantified phrases like *everything, each thing, all things*, and *anything*.
- In Logic, the symbol ∀ is used to express universal claims.
- It is always used in connection with a variable.
- It is a *variable binding* operator.
- The combination ∀x is read "for every object x"
- Somewhat misleadingly: "for all x"
- If we wanted to translate the (rather unlikely) English sentence *Everything is at home* into first-order logic, we would use the FOL sentence: ∀x Home(x)
- This says that every object x meets the following condition: x is at home.
- To put it more naturally, it says that everything whatsoever is at home.

Universal Quantifier (∀)

- Of course, we rarely make such unconditional claims about absolutely everything.
- More common are restricted universal claims like *Every doctor is smart*.
- This sentence would be translated as:

  $\forall x\ (Doctor(x) \rightarrow Smart(x))$
- This FOL sentence claims that given any object at all - call it x - if x is a doctor, then x is smart.
- To put it another way, the sentence says that if you pick anything at all, you'll find either that it is not a doctor or that it is smart (or perhaps both).

Existential Quantifier (∃)

- In English we have quantified phrases like *something, at least one thing, a*, and *an*.
- In Logic, the symbol ∃ is used to express existential claims.
- It is always used in connection with a variable
- It is a variable binding operator.
- The combination $\exists x$ is read *for some object x*.
- Somewhat misleadingly: *For some x.*
- If we wanted to translate the English sentence *Something is at home* into first-order logic, we would use the FOL sentence: $\exists x\ Home(x)$
- This says that some object x meets the following condition: x is at home.

Existential Quantifier (∃)

- While it is possible to make such claims, it is more common to assert that something of a particular kind meets some condition, say, *Some doctor is smart*.
- This sentence would be translated as:

  $\exists x$ (Doctor(x) ^ Smart(x))
- This sentence claims that some object, call it x, meets the complex condition: x is both a doctor and smart.
- More colloquially, it says that there is at least one smart doctor.

Free and Bound Variables

- A variable that occurs in an atomic wff is *free* or *unbound*.
- We use the quantifiers to bind variables.
- Any occurrence of the variable v is bound in the following two wffs:
  - $\forall v$ P
  - $\exists v$ P
- A variable may occur free or bound in a wff.
- Use parentheses to indicate scope of quantifiers.
- A *sentence* is a wff with **no free** variables.

Semantics for the Quantifiers

- Defined through “satisfaction.”

Satisfaction

- We say that an object satisfies the atomic wff Cube(x) if and only if the object is a cube.
- Similarly, we say an object satisfies the complex wff Cube(x) ^ Small(x) if and only if it is both a cube and small.

Semantics of ∃

- A sentence of the form ∃x S(x) is true if and only if **there is at least one** object that satisfies the constituent wff S(x).
- For example, ∃x (Cube(x) ^ Small(x)) is true if there is at least one object that satisfies Cube(x) ^ Small(x)
- In other words, there is at least one small cube.

Semantics of ∀

- A sentence of the form x ∀S(x) is true if and only if **every** object satisfies the constituent wff S(x).
- For example,∀x (Cube(x) → Small(x)) is true if every object satisfies Cube(x) → Small(x).
- In other words, every object either isn't a cube or it is small.

Domain of Discourse

- In general, sentences containing quantifiers are only true or false relative to some domain of discourse or domain of quantification.
- Sometimes the intended domain contains all objects there are.
- Usually, the intended domain is a much more restricted collection of things, say the people in the room, or some particular set of physical objects, or some collection of numbers.

Aristotelian Forms

- All P's are Q's.
- Some P's are Q's.
- No P's are Q's.
- Some P's are not Q's.

Aristotelian Forms

- All P's are Q's. $\forall x\ (P(x) \rightarrow Q(x))$
- Some P's are Q's.
- No P's are Q's.
- Some P's are not Q's.

Aristotelian Forms

- All P's are Q's. $\forall x\ (P(x) \rightarrow Q(x))$
- Some P's are Q's. $\exists x\ (P(x) \wedge Q(x))$
- No P's are Q's.
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- No P's are Q's. $\forall x\ (P(x) \rightarrow \neg Q(x))$
- Some P's are not Q's.

Aristotelian Forms

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- Some P's are not Q's. $\exists x\ (P(x) \wedge \neg Q(x))$

Translating Complex Noun Phrases

- *A small, happy dog is at home.*
- This sentence claims that there is an object which is simultaneously a small, happy dog, and at home.
- We would translate it as:

  $\exists$x [(Small(x) ^ Happy(x) ^ Dog(x)) ^ Home(x)]

Translating Complex Noun Phrases

- *Every small dog that is at home is happy*.
- This claims that everything with a complex property, that of being a small dog at home, has another property, that of being happy.
- We would translate it as:

  $\forall$x [(Small(x) ^ Dog(x) ^ Home(x)) $\rightarrow$ Happy(x)]
- In this case, the parentheses are not optional.
- Without them the expression would not be well formed.

Translating Complex Noun Phrases

- *Max owns a small, happy dog.*

$\exists$x [(Small(x) ^ Happy(x) ^ Dog(x)) ^ Owns(max, x)]

Translating Complex Noun Phrases

- *Max owns every small, happy dog.*

$\forall$x [(Small(x) ^ Happy(x) ^ Dog(x)) $\rightarrow$ Owns(max; x)]