$$(\forall x)\{P(x) \Rightarrow \{(\forall y)[P(y) \Rightarrow P(f(x,y))] \\ \wedge \sim(\forall y)[Q(x,y) \Rightarrow P(y)]\}\} .$$

The conversion process consists of the following steps:

(1) Eliminate implication symbols. All occurrences of the $\Rightarrow$ symbol in a wff are eliminated by making the substitution $\sim X1 \vee X2$ for $X1 \Rightarrow X2$ throughout the wff. In our example wff, this substitution yields:

$$(\forall x)\{\sim P(x) \vee \{(\forall y)[\sim P(y) \vee P(f(x,y))] \\ \wedge \sim(\forall y)[\sim Q(x,y) \vee P(y)]\}\} .$$

(2) Reduce scopes of negation symbols. We want each negation symbol, $\sim$, to apply to at most one atomic formula. By making repeated use of de Morgan's laws and other equivalences mentioned with them on pages 138–139, we change our example wff to:

$$(\forall x)\{\sim P(x) \vee \{(\forall y)[\sim P(y) \vee P(f(x,y))] \\ \wedge (\exists y)[Q(x,y) \wedge \sim P(y)]\}\} .$$

(3) Standardize variables. Within the scope of any quantifier, a variable bound by that quantifier is a dummy variable. It can be uniformly replaced by any other (non-occurring) variable throughout the scope of the quantifier without changing the truth value of the wff. Standardizing variables within a wff means to rename the dummy variables to ensure that each quantifier has its own unique dummy variable. Thus, instead of writing $(\forall x)[P(x) \Rightarrow (\exists x)Q(x)]$, we write $(\forall x)[P(x) \Rightarrow (\exists y)Q(y)]$. Standardizing our example wff yields:

$$(\forall x)\{\sim P(x) \vee \{(\forall y)[\sim P(y) \vee P(f(x,y))] \\ \wedge (\exists w)[Q(x,w) \wedge \sim P(w)]\}\} .$$

(4) Eliminate existential quantifiers. Consider the wff

$$(\forall y)[(\exists x)P(x,y)],$$

which might be read as "For all y, there exists an x (possibly depending on y) such that $P(x,y)$." Note that because the existential quantifier is within the scope of a universal quantifier, we allow the possibility that the x that exists might depend on the value of y. Let this dependence be explicitly defined by some function $g(y)$, which maps each value of y into the x that "exists." Such a function is called a ***Skolem function***. If we use the Skolem function in place of the x that exists, we can eliminate the existential quantifier altogether and write $(\forall y)P[g(y),y]$.

The general rule for eliminating an existential quantifier from a wff is to replace each occurrence of its existentially quantified variable by a Skolem function whose arguments are those universally quantified variables that are bound by universal quantifiers whose scopes include the scope of the existential quantifier being eliminated. Function symbols used in Skolem functions must be new in the sense that they cannot be ones that already occur in the wff. Thus, we can eliminate the $(\exists z)$ from

$$[(\forall w)Q(w)] \Rightarrow (\forall x)\{(\forall y)\{(\exists z)[P(x,y,z) \\ \Rightarrow (\forall u)R(x,y,u,z)]\}\},$$

to yield

$$[(\forall w)Q(w)] \Rightarrow (\forall x)\{(\forall y)[P(x,y,g(x,y)) \\ \Rightarrow (\forall u)R(x,y,u,g(x,y))] \ .$$

If the existential quantifier being eliminated is not within the scope of any universal quantifiers, we use a Skolem function of no arguments, which is just a constant. Thus, $(\exists x)P(x)$ becomes $P(A)$, where the constant symbol A is used to refer to the entity that we know exists. It is important that A be a new constant symbol and not one used in other formulas to refer to known entities.

To eliminate all of the existentially quantified variables from a wff, we use the above procedure on each formula in turn. Eliminating the existential quantifiers (there is just one) in our example wff yields:

$$(\forall x)\{\sim P(x) \vee \{(\forall y)[\sim P(y) \vee P(f(x,y))] \\ \wedge [Q(x,g(x)) \wedge \sim P(g(x))]\}\},$$

where $g(x)$ is a Skolem function.

(5) Convert to prenex form. At this stage, there are no remaining existential quantifiers and each universal quantifier has its own variable. We may now move all of the universal quantifiers to the front of the wff and let the scope of each quantifier include the entirety of the wff following it. The resulting wff is said to be in *prenex form*. A wff in prenex form consists of a string of quantifiers called a *prefix* followed by a quantifier-free formula called a *matrix*. The prenex form of our wff is:

$$(\forall x)(\forall y)\{\sim P(x) \vee \{[\sim P(y) \vee P(f(x,y))] \\ \wedge [Q(x,g(x)) \wedge \sim P(g(x))]\}\} .$$

(6) Put matrix in conjunctive normal form. Any matrix may be written as the conjunction of a finite set of disjunctions of literals. Such a matrix is said to be in *conjunctive normal form*. Examples of matrices in conjunctive normal form are:

$$[P(x) \vee Q(x,y)] \wedge [P(w) \vee \sim R(y)] \wedge Q(x,y)$$
$$P(x) \vee Q(x,y)$$
$$P(x) \wedge Q(x,y)$$
$$\sim R(y)$$

We may put any matrix into conjunctive normal form by repeatedly using one of the distributive rules, namely, by replacing expressions of the form $X1 \vee (X2 \wedge X3)$ by $(X1 \vee X2) \wedge (X1 \vee X3)$.

When the matrix of our example wff is put in conjunctive normal form, our wff becomes:

$$(\forall x)(\forall y)\{[\sim P(x) \vee \sim P(y) \vee P(f(x,y))] \\ \wedge [\sim P(x) \vee Q(x,g(x))] \wedge [\sim P(x) \vee \sim P(g(x))]\} .$$

(7) Eliminate universal quantifiers. Since all of the variables in the wffs we use must be bound, we are assured that all the variables remaining at this step are universally quantified. Furthermore, the order of universal quantification is unimportant, so we may eliminate the explicit occurrence of universal quantifiers and assume, by convention, that all variables in the matrix are universally quantified. We are left now with just a matrix in conjunctive normal form.

(8) Eliminate $\wedge$ symbols. We may now eliminate the explicit occurrence of $\wedge$ symbols by replacing expressions of the form $(X1 \wedge X2)$ with the set of wffs $\{X1, X2\}$. The result of repeated replacements is to obtain a finite set of wffs, each of which is a disjunction of literals. Any wff consisting solely of a disjunction of literals is called a *clause*. Our example wff is transformed into the following set of clauses:

$$\sim P(x) \vee \sim P(y) \vee P[f(x,y)]$$
$$\sim P(x) \vee Q[x,g(x)]$$
$$\sim P(x) \vee \sim P[g(x)]$$

(9) Rename variables. Variable symbols may be renamed so that no variable symbol appears in more than one clause. Recall that $(\forall x)[P(x) \wedge Q(x)]$ is equivalent to $[(\forall x)P(x) \wedge (\forall y)Q(y)]$. This process is sometimes called *standardizing the variables apart*. Our clauses are now:

$$\sim P(x1) \vee \sim P(y) \vee P[f(x1,y)]$$
$$\sim P(x2) \vee Q[x2,g(x2)]$$
$$\sim P(x3) \vee \sim P[g(x3)]$$

We note that the literals of a clause may contain variables but that these variables are always understood to be universally quantified. If terms not containing variables are substituted for the variables in an expression, we obtain what is called a *ground instance* of the literal. Thus, $Q(A,f(g(B)))$ is a ground instance of $Q(x,y)$.

Let us consider a simple example of this process. Suppose the following statements are asserted:

(1) Whoever can read is literate.
$(\forall x)[R(x) \Rightarrow L(x)]$

(2) Dolphins are not literate.
$(\forall x)[D(x) \Rightarrow \sim L(x)]$

(3) Some dolphins are intelligent.
$(\exists x)[D(x) \wedge I(x)]$

From these, we want to prove the statement:

(4) Some who are intelligent cannot read.
$(\exists x)[I(x) \wedge \sim R(x)]$

The set of clauses corresponding to statements 1 through 3 is:

(1) $\sim R(x) \vee L(x)$

(2) $\sim D(y) \vee \sim L(y)$

(3a) $D(A)$

(3b) $I(A)$

where the variables have been standardized apart and where A is a Skolem constant. The negation of the theorem to be proved, converted to clause form, is

(4′) $\sim I(z) \vee R(z)$.

To prove our theorem by resolution refutation involves generating resolvents from the set of clauses 1-3 and 4′, adding these resolvents to the set, and continuing until the empty clause is produced. One possible proof (there are more than one) produces the following sequence of resolvents:

(5) $R(A)$ resolvent of 3b and 4′

(6) $L(A)$ resolvent of 5 and 1

(7) $\sim D(A)$ resolvent of 6 and 2

(8) NIL resolvent of 7 and 3a