Completeness for FOL

Overview

- ✓ Adding Witnessing Constants
- ✓ The Henkin Theory
- The Elimination Theorem
- The Henkin Construction

Overview: Adding Witnessing Constants

- Let *L* be a fixed first-order language.
- We want to prove that if a sentence **S** of *L* is a first-order consequence of a set *T* of *L* sentences, then *T* |- **S**.
- The first step is to enrich *L* to a language L_H with infinitely many new constant symbols, known as *witnessing constants*, in a particular manner.

Overview: The Henkin Theory

- We next isolate a particular theory *H* in the enriched language L_H.
- This theory consists of various sentences which are not tautologies but are theorems of first-order logic, plus some additional sentences known as *Henkin witnessing axioms*.
- The latter take the form $\exists x P(x) \rightarrow P(c)$ where c is a witnessing constant.
- The particular constant is chosen carefully so as to make the Henkin Construction Lemma and Elimination Theorem true.

Lemma 7

- **Lemma 7**. Let *T* be a set of first-order sentences of some first-order language *L*, and let **P, Q**, and **R** be sentences of *L*.
  1. If T|- P → Q and T|- ¬P → Q then T|- Q.
  2. If T|- (P → Q) → R then T|- ¬P → R and T|- Q → R.
- **Proof:** (1) We have already seen that P V ¬P is provable without any premises at all.
- Hence, T |- P V ¬P.

Lemma 7

- Thus, our result will follow from Proposition 6 if we can show that the following argument has a proof in *F*:

$$P \lor \neg P$$
$$P \to Q$$
$$\neg P \to Q$$
$$\overline{\quad Q \quad}$$

- But this is obvious by ∨ Elim.
- The proof of (2) is similar, using Exercises 19.12 and 19.13.

Lemma 8

- Lemma 8 shows how certain constants in proofs can be replaced by quantifiers, using the rule of ∃ **Elim.**
- **Lemma 8.** Let *T* be a set of first-order sentences of some first-order language *L* and let **Q** be a sentence. Let P(x) be a wff of *L* with one free variable and which does not contain c. If *T* |- P(c) → Q and c does not appear in *T* or Q, then *T* |- ∃xP(x) → Q.

Proof of Lemma 8

- **Proof.** Assume that *T*|- P(c) → Q
- **c** is a constant that does not appear in *T* or Q.
- It is easy to see that for any other constant **d** not in *T* , P(x), or Q: *T* |- P(d) → Q.
- Just take the original proof *p* and replace **c** by **d** throughout
- If **d** happened to appear in the original proof, replace it by some other new constant, **c** if you like.
- Let us now give an informal proof, from *T* , of the desired conclusion ∃xP(x) → Q, being careful to do it in a way that is easily formalizable in *F*.

Proof of Lemma 8

- Using the method of $\rightarrow$ Intro, we take $\exists x\ P(x)$ as a premise and try to prove Q.
- We use the rule of $\exists$ Elim.
- Let **d** be a new constant and assume P(d).
- By our first observation, we know we can prove $P(d) \rightarrow Q$.
- By modus ponens ($\rightarrow$ Elim), we obtain Q as desired.
- This informal proof can clearly be formalized within F, establishing our result.

Lemma 9

- By combining Lemmas 7 and 8, we can prove Lemma 9, which is just what we need to eliminate the Henkin witnessing axioms.
- **Lemma 9.** Let *T* be a set of first-order sentences of some first-order language *L* and let Q be a sentence of *L*. Let P(x) be a wff of *L* with one free variable which does not contain **c**. If $T \cup \{\exists x P(x) \rightarrow P(c)\}$ |- Q and **c** does not appear in *T* or Q, then *T* |- Q.

Lemma 9

- **Proof.** Assume $T \cup \{\exists x P(x) \rightarrow P(c)\} \vdash Q$, where **c** is a constant that does not appear in *T* or Q.
- By the Deduction Theorem (Proposition 5):

  $$T \vdash (\exists x P(x) \rightarrow P(c)) \rightarrow Q$$

- By (2) of Lemma 7:

  $$T \vdash \neg \exists x P(x) \rightarrow Q$$

- and

  $$T \vdash P(c) \rightarrow Q$$

- From the latter, using (1) of Lemma 8, we obtain

  $$T \vdash \exists x P(x) \rightarrow Q.$$

- Then by (1) of Lemma 7, $T \vdash Q$.

Lemma 10

- Lemma 9 will allow us to eliminate the Henkin witnessing axioms from formal proofs.
- But what about the other sentences in H?
- In conjunction with Proposition 6, the next result will allow us to eliminate these as well.
- **Lemma 10.** Let *T* be a set of first-order sentences, let P(x) be a wff with one free variable, and let **c** and **d** be constant symbols. The following are all provable in *F*:

  $P(c) \rightarrow \exists x P(x)$

  $\neg \forall x\, P(x) \leftrightarrow \exists x\, \neg P(x)$

  $(P(c) \wedge c = d) \rightarrow P(d)$

  $c = c$

Proof of Lemma 10

- **Proof**. The only one of these that is not quite obvious from the rules of inference of *F* is the DeMorgan biconditional.
- We have given those proofs earlier in the term.

Reminder: Elimination Theorem

- **Proposition 4. (The Elimination Theorem)** Let *p* be any formal first-order proof with a conclusion **S** that is a sentence of *L* and whose premises are sentences $P_1, \ldots, P_n$ of *L* plus sentences from *H*. There exists a formal proof *p′* of **S** with premises $P_1, \ldots, P_n$ alone.

Proof of Elimination Theorem

- We have now assembled the tools we need to prove the Elimination Theorem.
- **Proof**. Let *k* be any natural number.
- Let *p* be any formal first-order proof of a conclusion in *L*:
  - all of whose premises are all either sentences of *L* or
  - sentences from *H*, and such that
  - there are at most *k* from *H*.
- We show how to eliminate those premises that are members of *H*.

Proof of Elimination Theorem

- The proof is by induction on k.
- The basis case is where $k = 0$.
- There is nothing to eliminate, so we are done with this case.

Proof of Elimination Theorem

- Let us assume the result for *k* and prove it for k + 1.
- The proof breaks into two cases:
- *Case 1*: At least one of the premises to be eliminated, say **P**, is of one of the forms mentioned in Lemma 10.
- Then **P** can be eliminated by Proposition 6 giving us a proof with at most *k* premises to be eliminated.
- We can do this by the induction hypothesis.

Proof of Elimination Theorem

- *Case 2:* All of the premises to be eliminated are Henkin witnessing axioms.
- The basic idea is to eliminate witnessing axioms introducing young witnessing constants before eliminating their elders.
- Pick the premise of the form $\exists x P(x) \rightarrow P(c)$ whose witnessing constant **c** is as young as any of the witnessing constants mentioned in the set of premises to be eliminated.
- That is, the date of birth *n* of **c** is greater than or equal to that of any of witnessing constants mentioned in the premises.
- This is possible since there are only finitely many such premises.

Proof of Elimination Theorem

- By the independence lemma, **c** is not mentioned in any of the other premises to be eliminated.
- Hence, **c** is not mentioned in any of the premises or in the conclusion.
- By Lemma 9, $\exists x P(x) \rightarrow P(c)$ can be eliminated.
- This gets us to a proof with at most *k* premises to be eliminated.
- We can do that by our induction hypothesis.

Henkin Construction

- Proposition 3 allows us to take any first-order structure for L and turn it into one for L_H that makes all the same sentences true.
- This gives rise to a truth assignment h to all the sentences of L_H that respects the truthfunctional connectives.
- Just assign TRUE to all the sentences that are true in the structure, FALSE to the others.
- The main step in the Henkin proof of the Completeness Theorem is to show that we can reverse this process.

Henkin Construction

- **Theorem** (Henkin Construction Lemma) Let h be any truth assignment for L_H that assigns TRUE to all the sentences of the Henkin theory H. There is a first-order structure $\mathfrak{M}_h$ such that $\mathfrak{M}_h \models S$ for all sentences S assigned TRUE by the assignment h.

Equivalence Classes

- Define a binary relation ≡ on the domain of $\mathfrak{M}$, i.e., the constants of L_H as follows:

  c ≡ d if and only if h(c = d) = TRUE.
- **Lemma 11**. The relation ≡ is an equivalence relation.
- **Proof.** This follows immediately from Exercise 19.20.
- From this it follows that we can associate with each constant **c** its equivalence class

  [c] = {d | c ≡ d}

Exercise 19.20

- Show that all sentences of the following forms are tautological consequences of *H:*
  1. $c = c$
  2. $c = d \rightarrow d = c$
  3. $(c = d \wedge d = e) \rightarrow c = e$
- Re 1: Follows from H4
- Re 3: Follows from application of H5
- Re 2: Follows from application of H5

Definition of $\mathfrak{M}_{\mathfrak{h}}$

- This allows us to define our desired first-order structure $\mathfrak{M}_{\mathfrak{h}}$:
- The domain *D* of our desired first-order structure $\mathfrak{M}_{\mathfrak{h}}$ is the set of all such equivalence classes.
- We let each constant **c** of L_H name its own equivalence class [c].
- To make the notation simpler, we assume that R is binary.
- Relation symbol **R** is the set

$$\{<[c], [d]> \mid h(R(c, d)) = TRUE\}$$