Let $\mathfrak{M}$ be a first-order structure with domain D. A *variable assignment* in $\mathfrak{M}$ is, by definition, some (possibly partial) function g defined on a set of variables and taking values in the set D. Thus, for example, if $D = \{a, b, c\}$ then the following would all be variable assignments in $\mathfrak{M}$:

1. the function g_1 which assigns b to the variable x
2. the function g_2 which assigns a, b, and c to the variables x, y, and z, respectively
3. the function g_3 which assigns b to all the variables of the language
4. the function g_4 which is the empty function, that is, does not assign values to any variables

The special case of the empty variable assignment g_4 is important, so we denote it by $g_\emptyset$.

Given a wff P, we say that the variable assignment g is *appropriate* for P if all the free variables of P are in the domain of g, that is, if g assigns objects to each free variable of P. Thus the four variable assignments g_1, g_2, g_3, and g_4 listed above would have been appropriate for the following sorts of wffs, respectively:

1. g_1 is appropriate for any wff with the single free variable x, or with no free variables at all;
2. g_2 is appropriate for any wff whose free variables are a subset of $\{x, y, z\}$;
3. g_3 is appropriate for any wff at all; and
4. g_4 (which we just agreed to write as $g_\emptyset$) is appropriate for any wff with no free variables, that is, for sentences, but not for wffs with free variables.

~~quantifier, we need a way to modify a variable assignment.~~ For example, if g is defined on x and we want to say what it means for g to satisfy $\forall$z Likes(x, z), then we need to be able to take any object b in the domain of discourse and consider the variable assignment which is just like g except that it assigns the value b to the variable z. We will say that g satisfies our wff $\forall$z Likes(x, z) if and only if every such modified assignment g' satisfies Likes(x, z). To make this a bit easier to say, we introduce the notation "$g[z/b]$" for the modified variable assignment. Thus, in general, $g[v/b]$ is the variable assignment whose domain is that of g plus the variable v and which assigns the same values as g, except that the new assignment assigns b to the variable v.

Here are a couple examples, harking back to our earlier examples of variable assignments given above:

1. g_1 assigns b to the variable x, so $g_1[\mathsf{y}/c]$ assigns b to x and c to y. By contrast, $g_1[\mathsf{x}/c]$ assigns a value only to x, the value c.

2. g_2 assigns a, b, c to the variables x, y, and z, respectively. Then $g_2[\mathsf{x}/b]$ assigns the values b, b, and c to x, y, and z, respectively. The assignment $g_2[\mathsf{u}/c]$ assigns the values c, a, b, and c to the variables u, x, y, and z, respectively.

3. g_3 assigns b to all the variables of the language. $g_3[\mathsf{y}/b]$ is the same assignment, g_3, but $g_3[\mathsf{y}/c]$ is different. It assigns c to y and b to every other variable.

4. g_4, the empty function, does not assign values to any variables. Thus $g_4[\mathsf{x}/b]$ is the function which assigns b to x. Notice that this is the same function as g_1.

Definition (Satisfaction) Let P be a wff and let g be an assignment in $\mathfrak{M}$ which is appropriate for P.

1. **The atomic case.** Suppose P is $R(t_1, \ldots, t_n)$, where R is an n-ary predicate. Then g satisfies P in $\mathfrak{M}$ if and only if the n-tuple $\langle [t_1]_g^{\mathfrak{M}}, \ldots, [t_n]_g^{\mathfrak{M}} \rangle$ is in $R^{\mathfrak{M}}$.
2. **Negation.** Suppose P is $\neg$Q. Then g satisfies P in $\mathfrak{M}$ if and only if g does not satisfy Q.
3. **Conjunction.** Suppose P is $Q \wedge R$. Then g satisfies P in $\mathfrak{M}$ if and only if g satisfies both Q and R.
4. **Disjunction.** Suppose P is $Q \vee R$. Then g satisfies P in $\mathfrak{M}$ if and only if g satisfies Q or R or both.
5. **Conditional.** Suppose P is $Q \rightarrow R$. Then g satisfies P in $\mathfrak{M}$ if and only if g does not satisfy Q or g satisfies R or both.
6. **Biconditional.** Suppose P is $Q \leftrightarrow R$. Then g satisfies P in $\mathfrak{M}$ if and only if g satisfies both Q and R or neither.
7. **Universal quantification.** Suppose P is $\forall v\, Q$. Then g satisfies P in $\mathfrak{M}$ if and only if for every $d \in D^{\mathfrak{M}}$, $g[v/d]$ satisfies Q.
8. **Existential quantification.** Suppose P is $\exists v\, Q$. Then g satisfies P in $\mathfrak{M}$ if and only if for some $d \in D^{\mathfrak{M}}$, $g[v/d]$ satisfies Q.

It is customary to write

$$\mathfrak{M} \models P\ [g]$$

to indicate that the variable assignment g satisfies wff P in the structure $\mathfrak{M}$.