Completeness of Propositional Logic – Part II

Lemma 5

- **Lemma 5.** A set of sentences is <u>formally complete</u> if and only if for every <u>atomic sentence</u> A, $T \vdash_T A$ or $T \vdash_T \neg A$
- **Proof.** The direction from left to right is just a consequence of the definition of formal completeness.

Lemma 5

- The direction from right to left is another example of a proof by induction on wffs.
- Assume that $T \vdash_T$ A or $T \vdash_T \neg$A for every atomic sentence A.
- We use induction to show that for any sentence S, $T \vdash_T$ S or $T \vdash_T \neg$S.
- The basis of the induction is given by our assumption.
- Let's prove the disjunction case.
- Assume S is of the form P v Q.
- By our inductive hypothesis, we know that *T* settles each of P and Q.
- If *T* proves either one of these, then we know that $T \vdash_T$ P v Q by v **Intro**.

Lemma 5

- Suppose that $T \vdash_T \neg P$ and $T \vdash_T \neg Q$. By merging these proofs and adding a step, we get a proof of $\neg P \wedge \neg Q$. We can continue this proof to get a proof of $\neg(P \vee Q)$, showing that $T \vdash_T \neg S$, as desired.
- The other inductive steps are similar.

Proposition 6

- **Proposition 6.** Every formally consistent set of sentences can be expanded to a formally consistent, formally complete set of sentences.
- **Proof.** Let us form a list A_1, A_2, A_3, ..., of all the atomic sentences of our language, say in alphabetical order.
- Go through these sentences one at a time.
- Whenever you encounter a sentence A_i such that neither A_i nor $\neg A_i$ is provable from the set, add A_i to the set.
- Doing so can't make the set formally inconsistent.

Proposition 6

- If you could prove $\bot$ from the new set, then you could prove $\neg A_i$ from the previous set, by Lemma 2.
- If that were the case, you wouldn't have thrown A_i into the set.
- The end result of this process is a set of sentences which is formally complete.
- It is also formally consistent.
- After all, any proof of $\bot$ is a finite object, and so could use at most a finite number of premises.
- In that case, we could have given a proof of $\bot$ at some point in the process of expanding the sentences.

Completeness

- **Theorem** (Completeness of F_T) If a sentence S is a tautological consequence of a set *T* of sentences then $T \vdash_{\mathrm{T}}$ S.
- **Proof**. Suppose $F \nvdash_{\mathrm{T}}$ S.
- By Lemma 2, $\mathrm{T} \cup \{\neg \mathrm{S}\} \nvdash_{\mathrm{T}} \perp$
- In other words, $\mathrm{T} \cup \{\neg \mathrm{S}\}$ is formally consistent.
- This set can be expanded to a formally consistent, formally complete set by Proposition 6.

Completeness

- By our Proposition 4, this set is tt-satisfiable.
- Suppose *h* is a truth value assignment that satisfies this set.
- *h* makes all the members of *T* true, but S false, because $T \cup \{\neg S\}$ is tt-satisfiable.
- But this means that S is not a tautological consequence of *T*, a contradiction.