Completeness of Propositional Logic

Definition. A set T of sentences is *tt-satisfiable* if there is a single assignment *h* that makes each of the sentences in T true.

Definition. A set T is *formally complete* if for **any** sentence S of the language, either T |-$_T$ S or T |-$_T$ ¬S.

Definition. A set of sentences T is *formally consistent* if and only T |/-$_T$ ⊥, that is, if and only if there is no proof of ⊥ from T in F_T.

Lemma 5. A set of sentences T is formally complete if and only if for every **atomic** sentence A, T |-$_T$ A or T |-$_T$ ¬A.

Proposition 6. Every formally consistent set of sentences T can be expanded to a formally consistent, formally complete set of sentences. (Use **Lemma 5**)

Lemma 3. Let T be a formally consistent, formally complete set of sentences, and let R and S be any sentences of the language. Then:

1. T |-$_T$ (R ^ S) iff T |-$_T$ R and T |-$_T$ S
2. T |-$_T$ (R v S) iff T |-$_T$ R or T |-$_T$ S
3. T |-$_T$ ¬S iff T |/-$_T$ S
4. T |-$_T$ (R → S) iff T |/-$_T$ R or T |-$_T$ S
5. T |-$_T$ (R ↔ S) iff either T |-$_T$ R and T |-$_T$ S or T |/-$_T$ R and T |/-$_T$ S

Proposition 4. Every formally consistent, formally complete set of sentences is tt-satisfiable. (Use **Lemma 3**)

Lemma 2. $T \cup \{\neg S\}$ |$-_T$ ⊥ if and only if T |$-_T$ S.

Theorem (Completeness of F_T) If a sentence S is a tautological consequence of a set T of sentences then T |$-_T$ S.

Proof. Suppose T |/-$_T$ S. Then by **Lemma 2**, $T \cup \{\neg S\}$ is formally consistent. This set can be expanded to a formally consistent, formally complete set by **Proposition 6**, which by our **Proposition 4** is tt-satisfiable. Suppose *h* is a truth value assignment that satisfies this set. Clearly, *h* makes all the members of T true, but S false, showing that S is not a tautological consequence of T.

Theorem (Reformulation of Completeness) Every formally consistent set of sentences is tt-satisfiable.