Completeness of Propositional Logic

Completeness of F_T

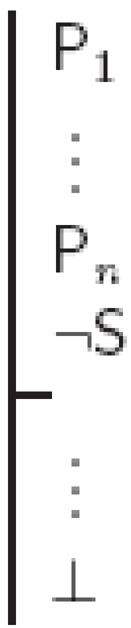
- **Theorem** (Completeness of F_T) If a sentence S is a tautological consequence of a set *T* of sentences then $T \vdash_T S$.

Lemma 2

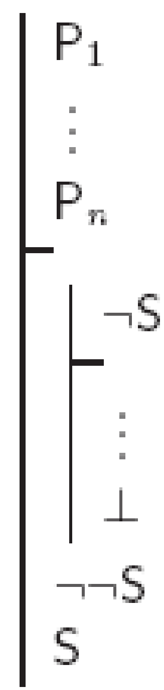
- **Lemma 2.** $T \cup \{\neg S\} \vdash_T \perp$ if and only if $T \vdash_T S$
- **Proof.** Assume $T \cup \{\neg S\} \vdash_T \perp$
- In other words, there is a proof of $\perp$ from premises $\neg S$ and certain sentences $P_1,\ldots,P_n$ of *T*.

Proof cont'd

Arrange the premises so that:



Rearrange to:



Other direction

- **Proof.** Assume $T \vdash_T S$
- In other words, there is a proof of S from certain sentences $P_1, \ldots, P_n$ of *T*.
- You finish!

Reformulating Completeness

- Lemma 2 shows that our assumption that $T \nvdash_{T} S$ is tantamount to assuming that $T \cup \{\neg S\} \nvdash \bot$
- **Definition.** A set of sentences is *formally consistent* if and only $T \nvdash_{T} \bot$, that is, if and only if there is no proof of $\bot$ from T in F_{T}.
- **Theorem** (Reformulation of Completeness) Every formally consistent set of sentences is tt-satisfiable.
- The Completeness Theorem results from applying this to the set $T \cup \{\neg S\}$.

Outline of Proof

- **Completeness for formally complete sets:** First we will show that the completeness theorem holds of any formally consistent set with an additional property, known as formal completeness.
  - **Definition.** A set T is *formally complete* if for any sentence S of the language, either $T \vdash_{T}$ S or $T \vdash_{T} \neg$S.
  - This means that the set T is so strong that it settles every question that can be expressed in the language.
  - In other words, for any sentence, either it or its negation is provable from T .

Lemma 3

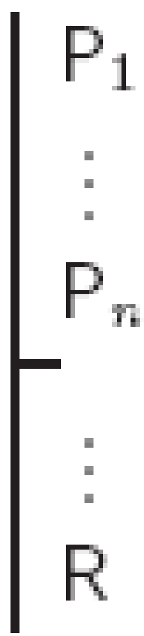
Lemma 3. Let T be a formally consistent, formally complete set of sentences, and let R and S be any sentences of the language.

1. $T \vdash_T (R \wedge S)$ iff $T \vdash_T R$ and $T \vdash_T S$
2. $T \vdash_T (R \vee S)$ iff $T \vdash_T R$ or $T \vdash_T S$
3. $T \vdash_T \neg S$ iff $T \nvdash S$
4. $T \vdash_T (R \rightarrow S)$ iff $T \nvdash_T R$ or $T \vdash_T S$
5. $T \vdash_T (R \leftrightarrow S)$ iff either $T \vdash_T R$ and $T \vdash_T S$ or $T \nvdash_T R$ and $T \nvdash_T S$

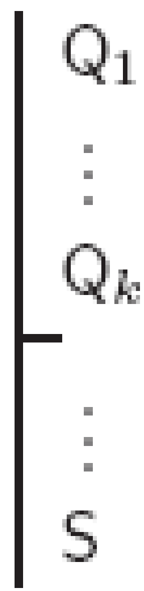
Proof of Lemma 3 (1)

- <u>Left-to-right</u>: Use ^ Elimination.
- <u>Right-to-left:</u> Proof by scissors:

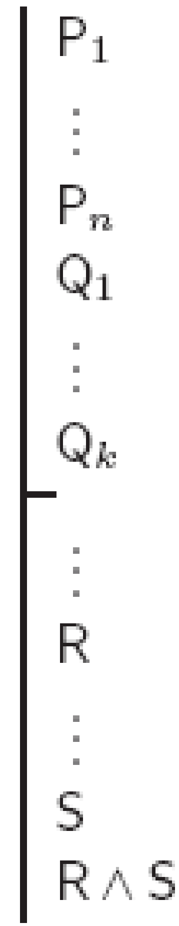
from:



and:



create:



Proof of Lemma 3 (2)

- Right-to-left: v-Introduction
- Left-to-right: […]

Proof of Lemma 3 (3)

- **Proof.** Left-to-right: By assumption, we can give a proof of ¬S from *T.* Suppose we can also give a proof of S. In that case *T* is not formally consistent, as we can give a proof of ⊥.

  Right-to-left: If we cannot give a proof of S from T, then by the definition of formally complete, we can give a proof of ¬S from *T.*

Proposition 4

- **Proposition 4.** Every formally consistent, formally complete set of sentences is tt-satisfiable.
- Lemma 3 tells us that we can give a proof for any sentence in a language that is formally complete and consistent.
- Proposition 4 tells us that we can find a truth function $\hat{h}$, making that set true.

Proposition 4

- **Proof**. Let *T* be the formally consistent, formally complete set of sentences.
- Define an assignment *h* on the atomic sentences of the language as follows.
- If T $\vdash_T$ A then let *h*(A) = TRUE; otherwise let *h*(A) = FALSE.
- The function $\hat{h}$ is defined on all the sentences of our language, atomic or complex.
- We claim that for all wffs S, $\hat{h}$(S) = TRUE if and only if T$\vdash_T$ S.

Proposition 4

- The proof of this is a good example of the importance of proofs by induction on wffs.
- The claim is true for all atomic wffs from the way that h is defined, and the fact that h and $\hat{h}$ agree on atomic wffs.
- We now show that if the claim holds of wffs R and S, then it holds of (R ∧ S), (R ∨ S), ¬R, (R → S) and (R ↔ S).
- These all follow easily from Lemma 3.

Proposition 4

- Consider the case of disjunction.
- We need to verify that $\hat{h}$ (R v S) = TRUE if and only if $T \vdash_{T}$ (R ∨ S).
- To prove the "only if" half, assume that $\hat{h}$ (R ∨ S) = TRUE.
- Then, by the definition of $\hat{h}$, either $\hat{h}$(R) = TRUE or $\hat{h}$ (S) = TRUE or both.
- Then, by the induction hypothesis, either $T \vdash_{T}$ R or $T \vdash_{T}$ S or both.
- But then by lemma 3, $T \vdash_{T}$ (R ∨ S), which is what we wanted to prove.
- The other direction is proved in a similar manner.