Substructural Logic

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What is Substructural Logic?

- A substructural logic is a non-classical logic that is weaker than classical logic in that it doesn't contain some of the structural rules present in classical logic.
- There are multiple substructural logics, depending on which principles are modified.
- These different modifications give rise to may different families of logics.

The Residuation Condition

- One of the main concepts of logic is the connection between logical consequence and the conditional termed the *residuation condition*, which states that

$$p\,,\, q \vdash r \;\Leftrightarrow\; p \vdash (q \rightarrow r)$$

- This is another way of restating the "conditional introduction" and "conditional elimination" rules of inference.
- Many substructural logics rise from different interpretations of the comma.

Weakening

- Occasionally, we may not want a logical system in which if p is true, then $q \rightarrow p$ is true. The proof for this would normally look like

$$p \vdash p$$

$$p\,,\, q \vdash p$$

$$p \vdash (q \rightarrow p)$$

- If we wish to reject the inference from p to $q \rightarrow p$, then we reject the reiteration step, we reject residuation, or we reject the first step of the proof. It is most plausible to consider the latter option.

Weakening (cont.)

- This first step is referred to as the *rule of weakening*:

$$X \vdash A \;\Rightarrow\; X, Y \vdash A$$

- It is rejected in a branch of substructural logic called *relevant logic*. In this logic, $p \rightarrow q$ is meant to say that q truly depends on p.
- We might have A following from X without A following from X, Y , since it need not be the case that A depends on X and Y together.
- Relevant logic becomes useful in subject areas such as philosophy.

Contraction

- Premises are often repeated in proofs. For example, consider the inference of $p \rightarrow q$ from $p \rightarrow (p \rightarrow q)$ in the proof

$$p \rightarrow (p \rightarrow q) \vdash p \rightarrow (p \rightarrow q)$$
$$p \rightarrow (p \rightarrow q)\ ,\ p \vdash (p \rightarrow q)$$
$$(p \rightarrow (p \rightarrow q)\ ,\ p)\ ,\ p \vdash q$$
$$p \rightarrow (p \rightarrow q)\ ,\ p \vdash q$$
$$p \rightarrow (p \rightarrow q) \vdash (p \rightarrow q)$$

- This proof uses only reiteration, residuation and a rule for repetition called the *rule of contraction*.

Contraction (cont.)

- The *rule of contraction* states that

$$X, X \vdash A \;\Leftrightarrow\; X \vdash A$$

- This rule and the rule of weakening are not used in *linear logic*. This logic is primarily useful in applications where the number of times premises are used must be monitored to maintain efficiency.
- This makes linear logic ideal as a way to model processing and resource use, as all resources must be used and resources cannot be used indefinitely.

Commutavity

- The commutativity of premises allows the inference of $(p \rightarrow q) \rightarrow q$ from p, evident from the proof:

$$p \rightarrow q \vdash p \rightarrow q$$
$$p \rightarrow q, p \vdash q$$
$$p, p \rightarrow q \vdash q$$
$$p \vdash (p \rightarrow q) \rightarrow q$$

- This conclusion is impossible to prove without the *rule of commutativity*, which states

$$X, Y \vdash A \Leftrightarrow Y, X \vdash A$$

Commutativity (cont.)

- In some instances, it is useful to reject the rule of commutativity.
- Suppose we wish the conditional to have modal force, which means $p \rightarrow q$ is meant to express that in every related circumstance in which p holds, q does too. Furthermore, we desire that logical consquence is meant to express local consequence, i.e. $p \vdash q$ if and only if in any model, at any circumstance at which p holds, so does q. Then this interpretation results in a rejection of commutativity.

Commutativity (cont.)

- To see this, consider the conclusion of the previous proof, $p \vdash (p \rightarrow q) \rightarrow q$.
- Using the previous interpretation, it may be true that Steve is a logician (p) and it is true that Steve's being a logician entails Steve's being a philosopher ($p \rightarrow q$ – in related circumstances in which Steve is a logician, he is a philosopher) but this does not entail that Steve is a philosopher, since there are many circumstances in which the entailment $p \rightarrow q$ is true but q is not.

Commutativity (cont.)

- The heart of this counterexample lies in the way the conditional has been restated.
- If we say $X \vdash A \rightarrow B$ (which by residuation is the same as $X , A \vdash B$), then we are saying that B holds in all circumstances that A holds, including the possibilities where X may not be true. This creates a distinction between $X , A \vdash B$ and $A , X \vdash B$.

Commutativity (cont.)

- The rejection of commutativity is useful in applications where order matters.
- This is the case in mathematical models of language and syntax, where the combinations of premises correspond to compositions of strings or other linguistic units. Not only does the number of premises used count, but also their order.
- This particular branch of logic is termed the Lambek Calculus, or also categorical grammar.

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