Proofs Involving Mixed Quantifiers

Good Proof

- Consider the following argument:

  $\exists y\ [\text{Girl}(y) \land \forall x\ (\text{Boy}(x) \rightarrow \text{Likes}(x, y))]$

  $\forall x\ [\text{Boy}(x) \rightarrow \exists y\ (\text{Girl}(y) \land \text{Likes}(x, y))]$

- **Proof:** Assume the premise.
- Thus, at least one girl is liked by every boy.
- Let *c* be one of these popular girls.

Good Proof

Girl(c) ∧ ∀x (Boy(x) → Likes(x, c))

∀x [Boy(x) → ∃y (Girl(y) ∧ Likes(x, y))]

- To prove the conclusion we will use general conditional proof.
- Assume that *d* is any boy in the class.
- We want to prove that *d* likes some girl.
- But every boy likes c, so d likes c.
- Thus d likes some girl, by existential generalization.
- Since d was an arbitrarily chosen boy, the conclusion follows.

Bad Proof

∀x [Boy(x) → ∃y (Girl(y) ∧ Likes(x, y))]

∃y [Girl(y) ∧ ∀x (Boy(x) → Likes(x, y))]

- **Pseudo-proof:** Assume the premise.
- Thus, every boy likes some girl or other.
- Let e be any boy in the domain.
- By our premise, e likes some girl.

Bad Proof

$\text{Boy}(e) \rightarrow \exists y\,(\text{Girl}(y) \wedge \text{Likes}(e, y))]$

$\forall x\,[\text{Boy}(x) \rightarrow \exists y\,(\text{Girl}(y) \wedge \text{Likes}(x, y))]$

- Let us introduce the new name f for some girl that e likes.
- Since the boy e was chosen arbitrarily, we conclude that every boy likes f, by general conditional proof.
- But then, by existential generalization, we have the desired result, namely, that some girl is liked by every boy.

Problems with Pseudo-proof

- The problem centers on our conclusion that every boy likes *f.*
- Recall how the name *f* came into the proof:
- We knew that *e*, being one of the boys, liked some girl.
- We chose one of those girls and dubbed her with the name *f.*
- Obviously, this choice of a girl depends crucially on which boy *e* we are talking about.
- If e was Matt or Alex, we could have picked Zoe and dubbed her *f.*
- But if e was Eric, we couldn't pick Zoe.
- Eric likes one of the girls, but certainly not Zoe.

Problems with Pseudo-proof

- The problem is this.
- Recall that in order to conclude a universal claim based on reasoning about a single individual, it is imperative that we not appeal to anything specific about that individual.
- But after we give the name *f* to one of the girls that *e* likes, any conclusion we come to about *e* and *f* may well violate this imperative.
- We can't be positive that it would apply equally to all the boys.

Problems with Pseudo-proof

- Stepping back from this particular example, the upshot is this.
- Suppose we assume P(c), where c is a new name, and prove Q(c).
- We cannot conclude $\forall x\ [P(x) \rightarrow Q(x)]$, if Q(c) mentions a specific individual whose choice depended on the individual denoted by c.
- In practice, the best way to insure that no such individual is specifically mentioned is to insist that Q(c) not contain *any* name that was introduced by existential instantiation under the assumption that P(c).

Another Pseudo-proof

- Assume ∀x ∃y Adjoins(x, y).
- Pseudo-proof: We will show ∃y ∀x Adjoins(x, y).
- We begin by taking c as a name for an arbitrary member of the domain.
- By universal instantiation, we get ∃y Adjoins(c, y).
- Let d be such that Adjoins(c, d).
- Since c stands for an arbitrary object, we have ∀x Adjoins(x, d).
- Hence, by existential generalization, we get ∃y ∀x Adjoins(x, y).