Formal Proofs and Quantifiers

∀ Elimination or Instantiation

- Suppose we are given as a premise that everything in the domain of discourse is either a cube or a tetrahedron.
- Suppose we also know that *c* is in the domain of discourse.
- It follows that c is either a cube or a tetrahedron, since everything is.
- More generally, suppose we have established $\forall x\ S(x)$, and we know that c names an object in the domain of discourse.
- We may legitimately infer $S(c)$.

∀ Elimination

Universal Elimination (∀ Elim):

$$\begin{array}{l|l} & \forall x\, S(x) \\ & \vdots \\ \triangleright & S(c) \end{array}$$

- x: any variable,
- c: any individual constant, whether used or not

∀ Introduction

- In formal systems of deduction, the method of general conditional proof is usually broken down into two parts:
  - Conditional proof and
  - A method for proving completely general claims, claims of the form ∀x S(x).
- The latter method is called universal generalization or universal introduction.
- It tells us that
  - if we are able to introduce a new name c to stand for a completely arbitrary member of the domain of discourse and
  - go on to prove the sentence S(c),
  - then we can conclude ∀x S(x).

∀ Introduction

- Here is a very simple example. Suppose we give an informal proof that the following argument is valid.

$\forall x\,(Cube(x) \rightarrow Small(x))$

$\forall x Cube(x)$

$\forall x Small(x)$

∀ Introduction

- Proof. We begin by taking a new name d, and think of it as standing for any member of the domain of discourse.
- Applying universal instantiation twice, once to each premise, gives us

  Cube(d) → *Small(d)*

  Cube(d)
- By modus ponens, we conclude Small(d).
- "d" denotes an arbitrary object in the domain
- Our conclusion, ∀x Small(x), follows by universal generalization.

$\forall$ Introduction

Universal Introduction ($\forall$ Intro):

$\boxed{c}$

$\vdots$

P(c)

$\triangleright$ $\forall x\, P(x)$

- c: does not occur outside the subproof where it is introduced.

Example of $\forall$ Intro and Elim

1. $\forall x\,(P(x) \rightarrow Q(x))$	
2. $\forall z\,(Q(z) \rightarrow R(z))$	
3. $\boxed{d}$ P(d)	
4. $P(d) \rightarrow Q(d)$	**$\forall$ Elim: 1**
5. Q(d)	**$\rightarrow$ Elim: 3, 4**
6. $Q(d) \rightarrow R(d)$	**$\forall$ Elim: 2**
7. R(d)	**$\rightarrow$ Elim: 5, 6**
8. $\forall x\,(P(x) \rightarrow R(x))$	**$\forall$ Intro: 3-7**

Class Exercise

13.6

$$\begin{array}{|l} \forall x\,((\mathrm{Cube}(x) \land \mathrm{Large}(x)) \\ \qquad \lor (\mathrm{Tet}(x) \land \mathrm{Small}(x))) \\ \forall x\,(\mathrm{Tet}(x) \to \mathrm{BackOf}(x, c)) \\ \forall x\,\neg(\mathrm{Small}(x) \land \mathrm{Large}(x)) \\ \hline \forall x\,(\mathrm{Small}(x) \to \mathrm{BackOf}(x, c)) \end{array}$$

(See Exercise 12.9. Notice that we have included a logical truth as an additional premise here.)

General Conditional Proof

- One of the most important methods of proof involves reasoning about an arbitrary object of a particular kind in order to prove a universal claim about all such objects.
- This is known as the method of *general conditional proof.*
- It is a more powerful version of conditional proof, and similar in spirit to the method of existential instantiation.

General Conditional Proof

- Let's start out with an example.
- Let us assume that the domain of discourse consists of students at a particular college.
- Suppose that we are given a bunch of information about these students in the form of premises.
- Let us suppose we are able to prove from these premises that Sandy, a math major, is smart.
- Under what conditions would we be entitled to infer that every math major at the school is smart?

General Conditional Proof

- At first sight, it seems that we could never draw such a conclusion, unless there were only one math major at the school.
- After all, it does not follow from the fact that one math major is smart that all math majors are.
- But what if our proof that Sandy is smart uses nothing at all that is particular to Sandy?
- What if the proof would apply equally well to any math major?
- Then it seems that we should be able to conclude that every math major is smart.

General Conditional Proof

- How might one use this in a real example?
- Let us suppose that our argument took the following form:

  Anyone who passes Logic 101 with an A is smart.
  Every math major has passed Logic 101 with an A.
  Every math major is smart.

General Conditional Proof

- Our reasoning proceeds as follows.
- Let "Sandy" refer to any one of the math majors.
- By the second premise, Sandy passed Logic 101 with an A.
- By the first premise, then, Sandy is smart.
- But since Sandy is an arbitrarily chosen math major, it follows that every math major is smart.

General Conditional Proof

- This method of reasoning is used at every turn in doing mathematics.
- The general form is the following.
- Suppose we want to prove ∀x [P(x) → Q(x)] from some premises.
- The most straightforward way to proceed is to choose a name that is not in use, say c, assume P(c), and prove Q(c).
- If you are able to do this, then you are entitled to infer the desired result.

General Conditional Proof

General Conditional Proof (∀ Intro):

$\boxed{c}$ $P(c)$

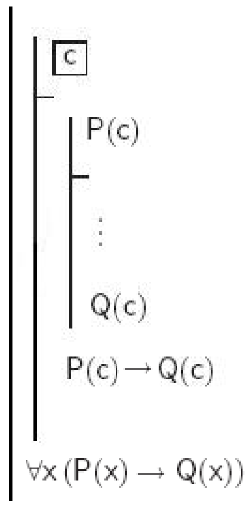
⋮

$Q(c)$

▷ $\forall x\,(P(x) \rightarrow Q(x))$

- c: does not occur outside the subproof where it is introduced.

Relationship between Universal Generalization and General Conditional Proof



∃ Introduction or Generalization

- Suppose you have established that *c* is a small tetrahedron.
- It follows that there is **some** small tetrahedron.
- There is no way for the specific claim about *c* to be true without the existential claim also being true.
- More generally, if we have established a claim of the form S(c) then we may infer ∃x S(x).

∃ Introduction

Existential Introduction (∃ Intro):

$$\begin{array}{l|l} & S(c) \\ & \vdots \\ \triangleright & \exists x\, S(x) \end{array}$$

- x: any variable
- c: individual constant or complex term without variables

∃ Elimination or Instantiation

- Existential instantiation is one of the more interesting and subtle methods of proof.
- It allows you to prove results when you are given an existential statement.
- Suppose our domain of discourse consists of all children, and you are told that some boy is at home.
- If you want to use this fact in your reasoning, you are of course not entitled to infer that Max is at home. Neither are you allowed to infer that John is at home.
- In fact, there is no particular boy about whom you can safely conclude that he is at home, at least if this is all you know.

∃ Elimination or Instantiation

- How should we proceed?
- We could give a temporary name to one of the boys who is at home, and refer to him using that name.
- We need to be careful not to use a name already used in the premises or the desired conclusion.

∃ Elimination or Instantiation

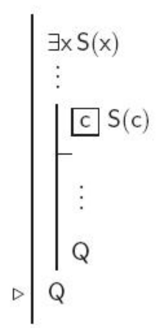
- This sort or reasoning is used in everyday life when we know that someone (or something) satisfies a certain condition, but do not know who (or what) satisfies it.
- For example, when Scotland Yard found out there was a serial killer at large, they dubbed him “Jack the Ripper,” and used this name in reasoning about him.
- No one thought that this meant they knew who the killer was; rather, they simply introduced the name to refer to whoever was doing the killing.

∃ Elimination or Instantiation

- Note that if the town tailor were already called Jack the Ripper, then the detectives' use of this name would (probably) have been a gross injustice.
- This is a basic strategy used when giving proofs in FOL.
- If we have correctly proven that ∃x S(x), then we can give a name, say c, to one of the objects satisfying S(x), as long as the name is not one that is already in use.
- We may then assume S(c) and use it in our proof.

∃ Elimination

Existential Elimination (∃ Elim):



- c: does not occur outside of the subproof where it is introduced.

Example of ∃ Intro and Elim

1.	$\forall x\,[\mathsf{Cube}(x) \rightarrow \mathsf{Large}(x)]$	
2.	$\forall x\,[\mathsf{Large}(x) \rightarrow \mathsf{LeftOf}(x, b)]$	
3.	$\exists x\,\mathsf{Cube}(x)$	
4.	$\boxed{e}\ \mathsf{Cube}(e)$	
5.	$\mathsf{Cube}(e) \rightarrow \mathsf{Large}(e)$	$\forall$ **Elim**: 1
6.	$\mathsf{Large}(e)$	$\rightarrow$ **Elim**: 5, 4
7.	$\mathsf{Large}(e) \rightarrow \mathsf{LeftOf}(e, b)$	$\forall$ **Elim**: 2
8.	$\mathsf{LeftOf}(e, b)$	$\rightarrow$ **Elim**: 7, 6
9.	$\mathsf{Large}(e) \land \mathsf{LeftOf}(e, b)$	$\land$ **Intro**: 6, 8
10.	$\exists x\,(\mathsf{Large}(x) \land \mathsf{LeftOf}(x, b))$	$\exists$ **Intro**: 9
11.	$\exists x\,(\mathsf{Large}(x) \land \mathsf{LeftOf}(x, b))$	$\exists$ **Elim**: 3, 4-10

Class Exercise

13.29 ★

$\forall x\,(\mathsf{Small}(x) \rightarrow \mathsf{Cube}(x))$

$\exists x\,\neg\mathsf{Cube}(x) \rightarrow \exists x\,\mathsf{Small}(x)$

$\exists x\,\mathsf{Cube}(x)$

Strategy and Tactics

- Always be clear about the meaning of the sentences you are using.
- A good strategy is to find an informal proof and then try to formalize it.
- Working backwards can be very useful in proving universal claims, especially those of the form $\forall x\ (P(x) \rightarrow Q(x))$.
- Working backwards is not useful in proving an existential claim $\exists x\ S(x)$ unless you can think of a particular instance $S(c)$ of the claim that follows from the premises.
- If you get stuck, consider using proof by contradiction.

Soundness and Completeness

- **Theorem** (Soundness of *F*) If $\mathcal{T} \vdash$ S, then S is a first-order consequence of set $\mathcal{T}$.
- **Theorem** (Completeness Theorem for *F*). Let $\mathcal{T}$ be a set of sentences of a first-order language *L* and let S be a sentence of the same language. If S is a first-order consequence of $\mathcal{T}$, then $\mathcal{T} \vdash$ S.