The Logic of Quantifiers

First-Order Logic

- In FOL, we are allowed to say things like *∃x Large(x),* that is, *there is something that has the property of being large.*
- We can't say things like there is some property that Max has:
  ∃P P(max).
- First-order quantifiers allow us to make quantity claims about ordinary objects: blocks, people, numbers, sets, and so forth.
- (Note that we are very liberal about what an ordinary object is.)
- We might be interested in making quantity claims about **properties** of the objects in our domain of discourse.
- For example, we may want to claim that Max and Claire share exactly two properties.
- In this case, we need what is known as *second-order quantifier.*
- Since our language only has first-order quantifiers, it is known as the language of first-order logic.

Tautologies and Quantification

- Are sentences valid or a tautology simply based on the truth-functional connectives?
- In other words, could we ignore the quantifiers?

Tautological Consequence and Quantification

- For example, the following are valid:

| ∀x (Cube(x) -> Small(x))
| ∀x Cube(x)
|—
| ∀x Small(x)

| ∀x Cube(x)
| ∀x Small(x)
|—
| ∀x (Cube(x) ^ Small(x))

Tautological Consequence and Quantification

- The following are **not** valid. Why not?

  | ∃x (Cube(x) -> Small(x))
  | ∃x Cube(x)
  |—
  | ∃x Small(x)

  | ∃x Cube(x)
  | ∃x Small(x)
  |—
  | ∃x (Cube(x) ^ Small(x))

Tautologies and Quantification

- Logically true:
  ∃x Cube(x) v ∃x ¬Cube(x))
- However, it is true not based on the meaning of the truth-functional connectives, consider:
  ∀x Cube(x) v ∀x ¬Cube(x))
- We cannot ignore quantifiers!
- Done right:
  ∀x Cube(x) v ¬∀x Cube(x))

Substitution of Sentences in Tautologies

- Consider the following tautology:
  $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
- If we replace A and B by any quantified sentences of FOL, the result is still a tautology.
- For example:
  A: $\exists y\ (P(y) \vee R(y))$ B: $\forall x\ (P(x) \wedge Q(x))$
- Gives:
  $(\exists y\ (P(y) \vee R(y)) \rightarrow \forall x\ (P(x) \wedge Q(x))) \rightarrow$
  $(\neg \forall x\ (P(x) \wedge Q(x)) \rightarrow \neg \exists y\ (P(y) \vee R(y)))$

Truth-functional Form Algorithm

1) Start at the beginning of sentence S and proceed to the right.
2) When you come to a quantifier or an atomic sentence, begin to underline that portion of the sentence.
   - 2.1) If you encountered a quantifier, underline the quantifier and the entire formula that it is applied to.
     - 2.1.1) It is either an atomic wff that immediately follows the quantifier
     - 2.1.2) Or, if there are parentheses, the formula enclosed by the parentheses.
   - 2.2) If you encountered an atomic sentence, just underline the atomic sentence.

Truth-functional Form Algorithm

3) When you come to the end of your underline, assign the underlined constituent a sentence letter (A, B, C, . . .).
   3.1) If an identical constituent already appears earlier in the sentence, use the same sentence letter as before.
   3.2) Otherwise, assign the first sentence letter not yet used as a label.
3) Once you've labeled the constituent, go back to step 2.
4) When you come to the end of the sentence, replace each underlined constituent with the sentence letter that labels it.

- The result is the truth-functional form of S.

Tautologies of FOL

- Use the truth-functional form algorithm to determine the truth-functional form of a sentence or argument containing quantifiers.
- The truth-functional form of a sentence shows how the sentence is built up from atomic and quantified sentences using truth-functional connectives.
- **Definition** A quantified sentence of FOL is said to be a *tautology* if and only if its truth-functional form is a tautology.
- Every tautology is a logical truth, but among quantified sentences there are many logical truths that are not tautologies.
- Similarly, there are many logically valid arguments of FOL that are not tautologically valid.

First-order validity and consequence

General Notion	Propositional Logic	First-Order Logic
Logical truth	Tautology	FO validity
Logical consequence	Tautological consequence	FO consequence
Logical equivalence	Tautological equivalence	FO equivalence

FO Validity

- If we can recognize that a sentence is logically true without knowing the meanings of the names or predicates it contains (other than identity), then we'll say the sentence is a *first-order validity.*
- Examples to consider:
  - ∀x SameSize(x, x)
  - ∀x Cube(x) -> Cube(b)
  - (Cube(b) ^ b = c) -> Cube(c)
  - (Small(b) ^ SameSize(b, c)) -> Small(c)

FO Validity

- Examples translated:
  - $\forall$x Foo(x, x)
  - $\forall$x Foo(x) -> Foo(b)
  - (Foo(b) ^ b = c) -> Foo(c)
  - (Foo(b) ^ Foobar(b, c)) -> Foo(c)
- Second and third sentences must be true, whatever the predicate *Foo* may mean.

FO consequence

- Use a similar process.
- Consider:

  $\forall$x (Tet(x) -> Large(x))
  ¬Large(b)
  ¬Tet(b)
- Translate to:

  $\forall$x (Foo(x) -> Foobar(x))
  ¬Foobar(b)
  ¬Foo(b)
- Conclusion is not just a logical consequence, but an FO consequence.

FO Counterexamples

- Consider the following set of sentences:

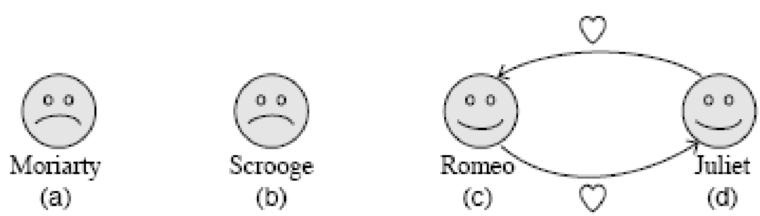
  ¬∃x Larger(x, a)
  ¬∃x Larger(b, x)
  Larger(c, d)
  Larger(a, b)

- Translate:

  ¬∃x Foo(x, a)
  ¬∃x Foo(b, x)
  Foo(c, d)
  Foo(a, b)

- Is the conclusion a consequence? Is it a logical consequence?

FO Counterexamples



- Let “Foo” be “Likes”

Replacement Method

To check for *first-order validity* or *first-order consequence*, systematically replace all of the predicates, other than identity, with new, meaningless predicate symbols, making sure that if a predicate appears more than once, you replace all instances of it with the same meaningless predicate. (If there are function symbols, replace these as well.)

Replacement Method

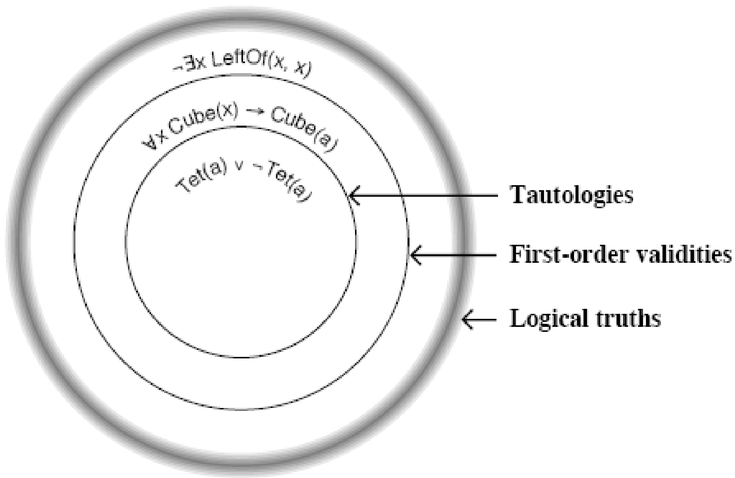
a) To see if S is a *first-order validity*, try to describe a circumstance, along with interpretations for the names, predicates, and functions in S, in which the sentence is false. If there is no such circumstance, the original sentence is a first-order validity.
b) To see if S is a *first-order consequence* of $P_1,\ldots, P_n$, try to find a circumstance and interpretation in which S is false while $P_1,\ldots,P_n$ are all true. If there is no such circumstance, the original inference counts as a first-order consequence.

- Recognizing whether a sentence is a first-order validity, or a first-order consequence of some premises, is not as routine as with tautologies and tautological consequence.
- With truth tables, there may be a lot of rows to check, but at least the number is finite and known in advance.
- With first-order validity and consequence, the situation is much more complicated, since there are infinitely many possible circumstances that might be relevant. In fact, there is no correct and mechanical procedure, like truth tables, that always answers the question is S a first-order validity?

Relationship between Tautologies, FO Validities and Logical Truths

- If S is a tautology, then it is a first-order validity.
- If S is a first-order validity, it is a logical truth.
- The converse of neither of these statements is true (see Figure on next slide.)
- If S is a tautological consequence of premises $P_1, \ldots,$ P_n, then it is a first-order consequence of these premises.
- If S is a first-order consequence of premises $P_1, \ldots, P_n$, then it is a logical consequence of these premises.
- The converse of neither of these previous two statements is true.

Relationship between Tautologies, FO Validities and Logical Truths



Remember

- A sentence of FOL is a first-order validity if it is a logical truth when you ignore the meanings of the names, function symbols, and predicates other than the identity symbol.
- A sentence S is a first-order consequence of premises P_1, …, P_n if it is a logical consequence of these premises when you ignore the meanings of the names, function symbols, and predicates other than identity.
- The Replacement Method is useful for determining whether a sentence is a first-order validity and whether the conclusion of an argument is a first-order consequence of the premises.