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Intuitionistic Logic

Outline

- Introduction
- Syntax and Semantics
- Rules of Inference
- Examples
- Completeness and Soundness
- Evaluating the System

Why is it interesting?

- x, y ∈ {0, 1, 2, 3, …}
- B(x): there exists a y > x such that both y and y+2 are prime numbers
- A: $\forall x B(x)$
- A v ¬A cannot be asserted because neither A nor (¬A) has yet been proven

Syntax and Semantics

- Similar to FOL
- "T" means that the statement has actually been proven

Rules of Inference

- Modus Ponens
- $\vee$ Intro, Elim
- $\wedge$ Intro, Elim
- $\longrightarrow$ Intro, Elim
- $\perp$ Intro, Elim
- Existential Intro, Elim
- Universal Intro, Elim

Axioms

- $A \longrightarrow (B \longrightarrow A)$
- $(A \longrightarrow B) \longrightarrow ((A \longrightarrow (B \longrightarrow C)) \longrightarrow (A \longrightarrow C))$
- $A \longrightarrow (B \longrightarrow A \wedge B)$
- $(A \longrightarrow C) \longrightarrow ((B \longrightarrow C) \longrightarrow (A \vee B \longrightarrow C))$
- $(A \longrightarrow B) \longrightarrow ((A \longrightarrow \neg B) \longrightarrow \neg A)$
- $\neg A \longrightarrow (A \longrightarrow B)$

Axioms and Laws

- Not necessarily true in intuitionistic logic
  - $P \vee \neg P$
  - $((P \rightarrow Q) \rightarrow P) \rightarrow P)$
  - $\neg\neg P \rightarrow P$
  - $\neg(\neg P \wedge \neg Q) \rightarrow (P \vee Q)$
  - $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$

Relation to Classical Logic

- To get back to classical logic, add one of:
  - $\phi \vee \neg\phi$
  - $\neg\neg\phi \rightarrow \phi$
  - $((\phi \rightarrow \chi) \rightarrow \phi) \rightarrow \phi$

Example Proof

- $\neg\neg(P \vee \neg P)$

Example – Failed Proof

- $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$

$\neg Q \rightarrow \neg P$

$\neg\neg Q \vee \neg P$

$Q \vee \neg P$

$P \rightarrow Q$

$(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$

Completeness & Soundness

- Completeness
  - Pure intuitionistic logic is axiomatically incomplete
- Soundness
  - Intuitionistic logic is sound

Strengths and Weaknesses

- Strengths
  - Disjunction and Existence Properties
  - Used in infinite situations because LEM is not included
- Weaknesses
  - Incomplete
  - Not every propositional formula has an intuitionistically equivalent disjunctive or conjunctive normal form
  - Not every predicate formula has an intuitionistically equivalent prenex form

References

- Mints, Grigori. A Short Introduction to Intuitionistic Logic. Kluwer Academic (2000).
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- Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/entries/logic-intuitionistic/