DIAGRAMMATIC REASONING AND LOGIC

DIAGRAMMATIC REASONING

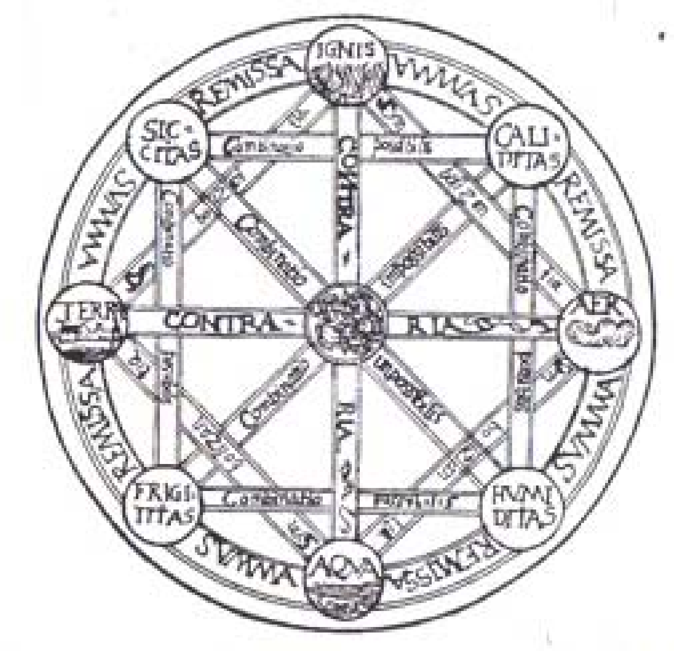
- Area of research concerned with how humans or machines can represent information using diagrams, then reason using those diagrams
- Diagram—pictorial yet abstract representation of information
- Examples include maps, line graphs, bar charts, architect's sketches, etc.

WHY STUDY DIAGRAMMATIC REASONING?

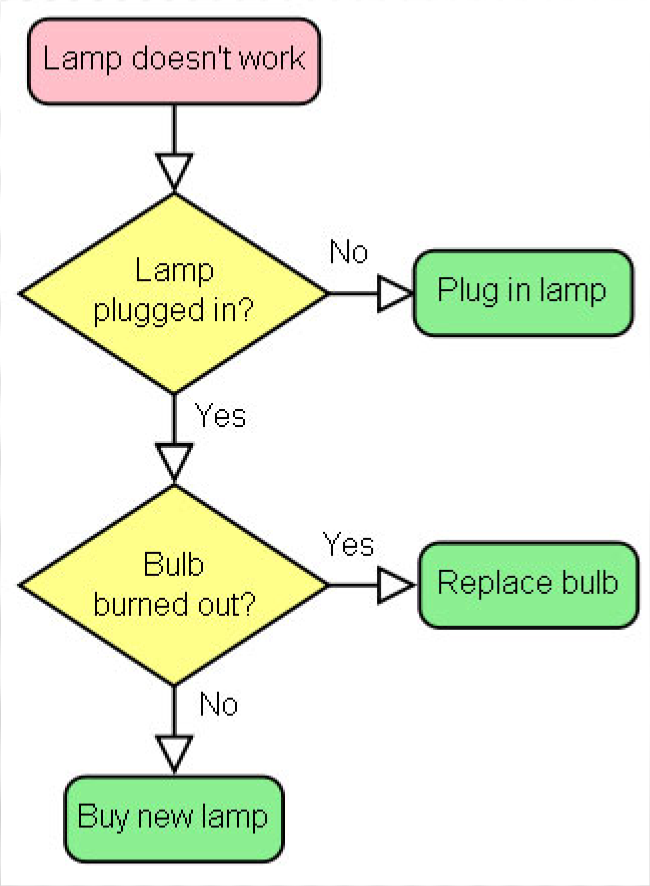
- Diagrams don’t have the rich detail of photographic depictions, but still capture and convey necessary information
- Can’t develop a syntax or semantics for photographs—only diagrams
- Able to reason from a visual representation of information

CHARACTERISTICA UNIVERSALIS

- Universal and Formal language
- Developed by Gottfried Leibniz in the late 1600s
- Able to express mathematical, scientific, and metaphysical concepts

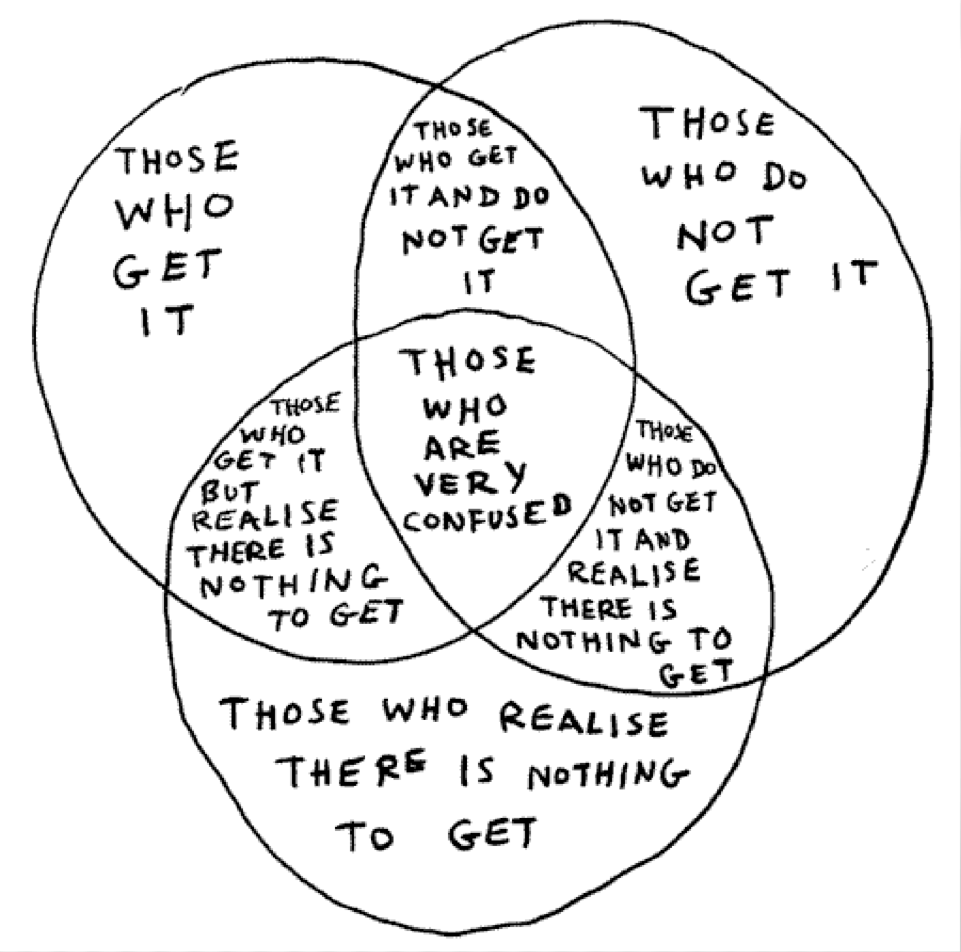


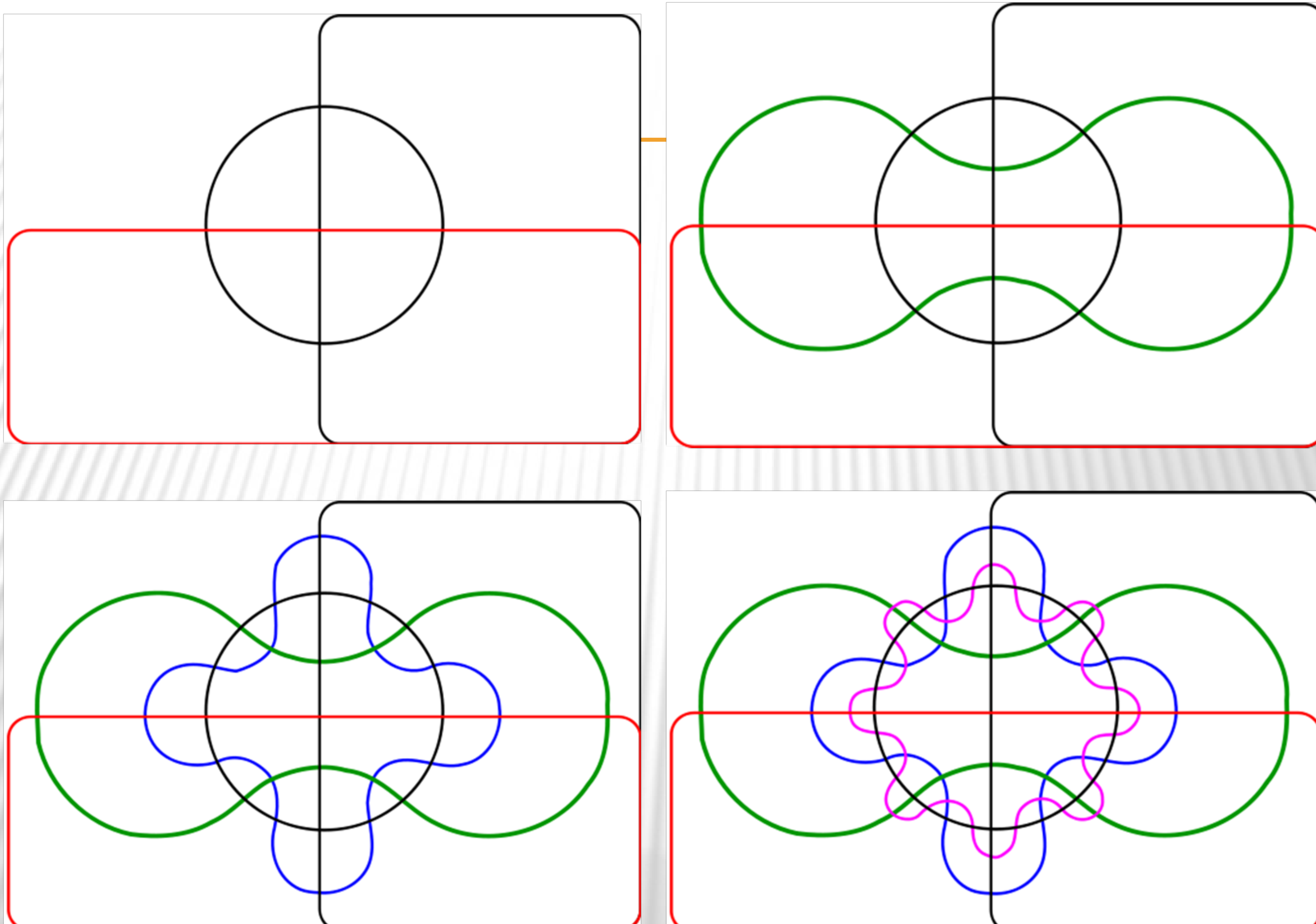
DIAGRAMS



- 2D symbolic representation of information according to some visualization technique
- Don't show quantitative data—show relationships and abstract information instead
- Use shapes connected by arrows, lines, and other visual links

VENN DIAGRAMS

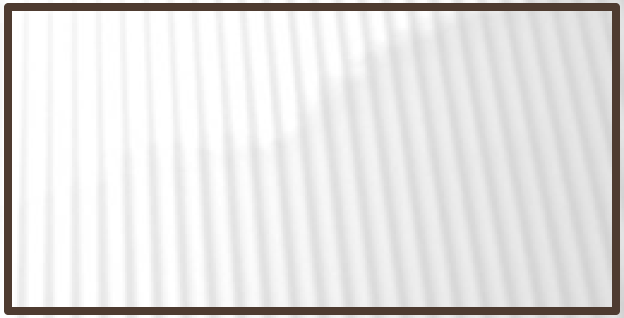


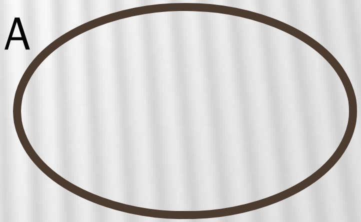


SYNTAX

BASIC ELEMENTS

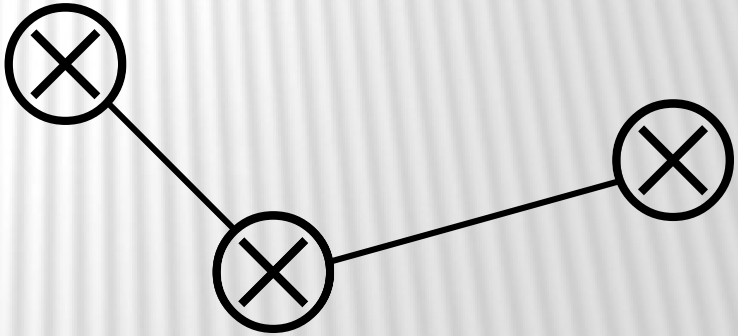
- Rectangle
  - Represents the domain of discourse
- Closed Curve
  - No self-intersection
  - Represent subsets of the domain
- Labels
  - Tag closed curves
  - Show when two closed curves intentionally represent the same set



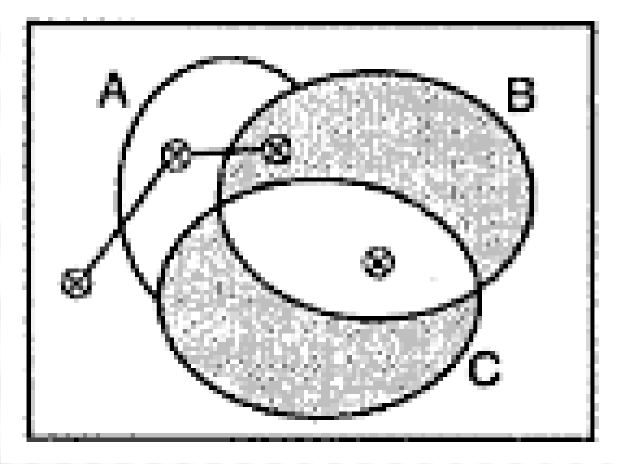


BASIC ELEMENTS

- Shading
  - Represents emptiness of a represented set
- ⊗ symbol
- Lines
  - ⊗ symbols connected by lines indicate the non-emptiness of a represented set



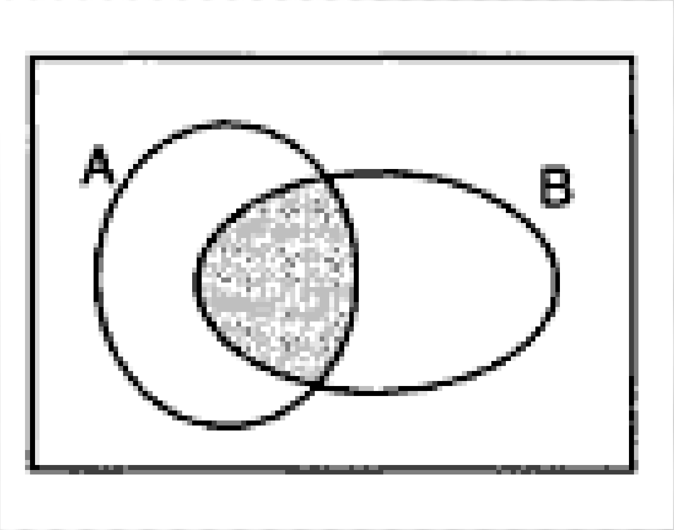
DIAGRAMS



- Consists of a rectangle within which are drawn various closed curves as well as shading, symbols, and lines

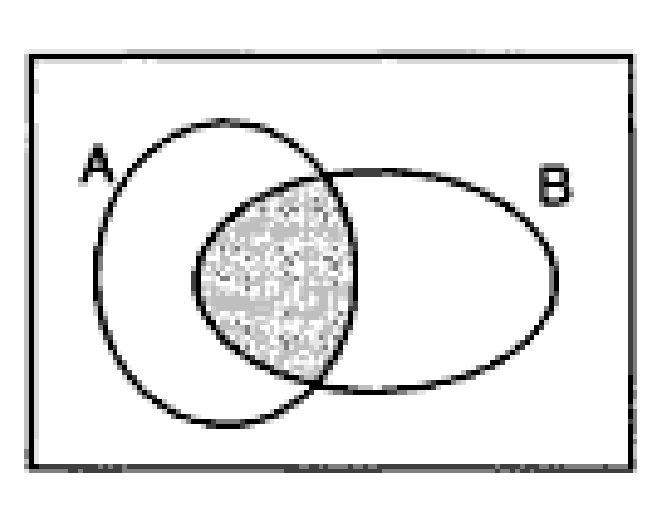
REGIONS

- A **basic region** is the area enclosed by the rectangle of D or any closed curve of D
- If r and s are regions of D, then so are:
  - R v S
  - R ^ S
  - The area included in r but not in s (represented by r\s)



REGIONS

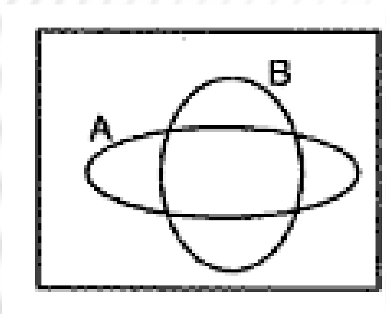
- Minimal Regions
  - A region r of a diagram D is a "minimal region" if and only if for all regions s, if s is a subregion of r, then s = r
  - Regions which have no other regions as proper parts

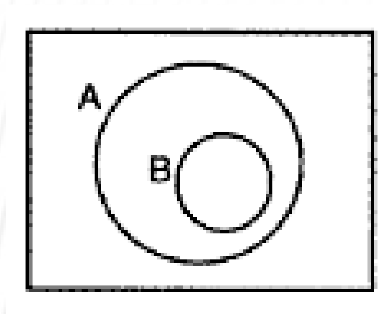


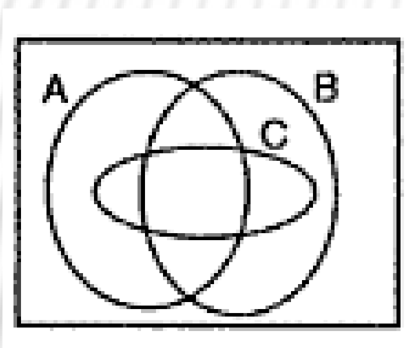
WELL-FORMED DIAGRAMS

- Any single drawn rectangle is a well-formed diagram

WELL-FORMED DIAGRAMS

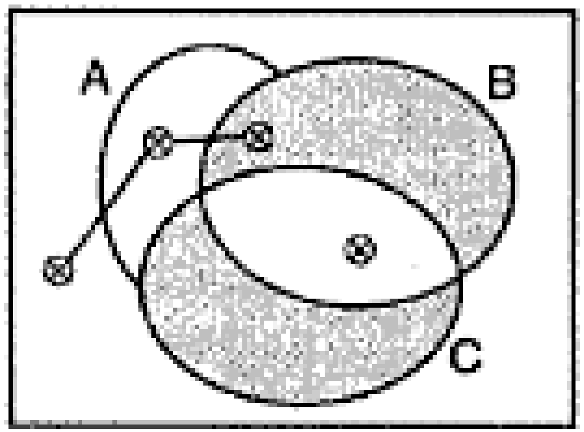


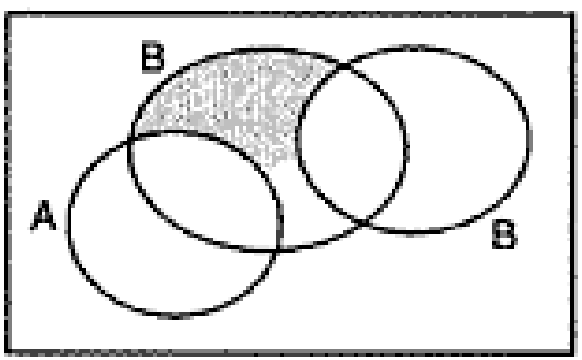




- If D is a well formed diagram and D’ results from D by adding a closed curve tagged by some label not occurring in D and added according to the “partially overlapping rule”, then D’ is a well-formed diagram.
- The partial overlapping rule demands that the new curve overlaps a proper part of every minimal region of D once and only once

WELL-FORMED DIAGRAMS

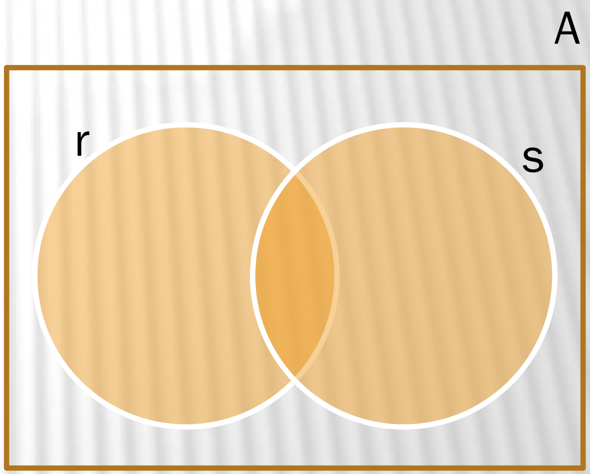




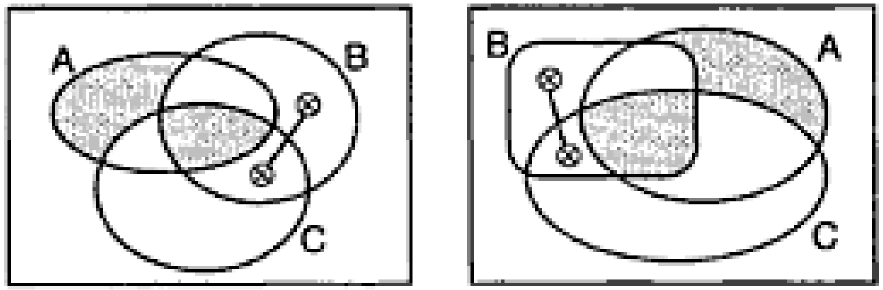
- If D is a well-formed diagram and D’ is obtained from D by shading some entire minimal region of D, then D’ is a well-formed diagram.
- If D is a well-formed diagram and D’ is obtained from D by adding a chain of one or more ⊗ connected by lines (an “⊗ -sequence” to D such that each ⊗ of the chain falls within a distinct minimal region and no other chain of ⊗’s in D has its ⊗’s fall in exactly those minimal regions, then D’ is a well-formed diagram.

COUNTERPART RELATIONS

- If r and s are basic regions enclosed by closed curves tagged by the same label, then r and s are counterparts.
- If r and s are basic regions enclosed by rectangles, then r and s are counterparts
- If r_1 and s_1 are counterparts and r_2 and s_2 are counterparts where r_1 and r_2 are regions of the same diagram and s_1 and s_2 are regions of the same diagram, then so are $(r_1 \vee r_2)$ and $(s_1 \vee s_2)$, $(r_1 \wedge r_2)$ and $(s_1 \wedge s_2)$, and $(r_1 \setminus r_2)$ and $(s_1 \setminus s_2)$



INSTANCES OF THE SAME DIAGRAM



- Two tokens of diagrams are "instances of the same diagram" or "tokens of the same diagram" if and only if the set of tags of the two diagrams are identical, and whenever two regions r and s of the two tokens are counterparts, then:
  - Region r is shaded if and only if s is shaded
  - Region r has an ⊗-sequence if and only if s has an ⊗-sequence

SEMANTICS

MODELS

- A **model** consists of a set along with and interpretation function assigning subsets of the set to regions of diagrams in a way that respects the intuitive meaning of the overlap of two regions, etc.
- Defined in two parts, an assignment function and the set to be assigned

MODELS

- Assignment Functions
  - If U is a set, call a function v assigning subsets of U to basic regions an “assignment function on U” if and only if:
    - Every basic region consisting of the area inside a rectangle is assigned the set U, and
    - If r is a region enclosed by a closed curve, then $v(r) \subseteq U$, with the restriction that if r and s are any two regions enclosed by closed curves tagged by the same label, then $v(r) = v(s)$

MODELS

- Let v be an assignment function on a set U. Then there is exactly one function I assigning subsets to regions such that for all basic regions r, I(r) = v(r), and for all regions r and s of a diagram D:
  - I(r v s) = I(r) v I(s)
  - If there is a region r^s, I(r^s) = I(r) ^ I(s)
  - If there is a region r\s, I(r\s) = I(r) \ I(s)

MODELS

- A model is defined by a pair M=(U,I)
  - Set U
  - Interpretation function I such that I is the unique extension of some assignment function v on the set U
- If regions r and s are counterparts and (U,I) is a model, then I(r) = I(s)

HOW IS A DIAGRAM "TRUE" IN A MODEL?

- Let M = (U,I) be a model and D be a diagram. Then D is "true in M" or "M models D" if and only if for each region r of D:
  - If r is shaded, then I(r)=0
  - If an ⊗-sequence occurs in r, then I(r) is not 0.

RULES OF INFERENCE AND PROOFS

RULES OF INFERENCE

- Erasure of part of an ⊗-sequence
  - D’ is obtainable from D if and only if D’ results from D by the erasure of some link of an ⊗-sequence that falls in a shaded region of D, provided the two halves of the sequence are reconnected by a line
- Spreading ⊗’s
  - D’ is obtainable from D if and only if D’ results from the addition of a new link to any sequence of D.

RULES OF INFERENCE

- Erasure of a diagrammatic object
  - A diagram D’ is obtainable from D if and only if D’ results from the erasure of any entire ⊗-sequence, or from the erasure of the shading of any region, or by the erasure of any closed curve in accordance with the following condition: the shading of any minimal region of D that would fail to cover an entire minimal region upon erasure of the curve must also be erased, and if the erasure of the curve would result in some ⊗-sequence having two links in some minimal region of D’, then one of those links must be erased (and the two halves rejoined)

RULES OF INFERENCE

- Introduction of new basic regions
  - D can be asserted at any link if D consists of a rectangle within which is a single closed curve but no shading or sequences, or else D consists of a single rectangle
- Conflicting Information
  - D' is obtainable from D if D' is any diagram and D has a region that is both shaded and has an ⊗-sequence

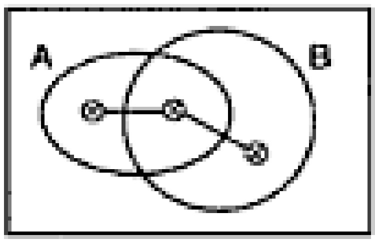
RULES OF INFERENCE

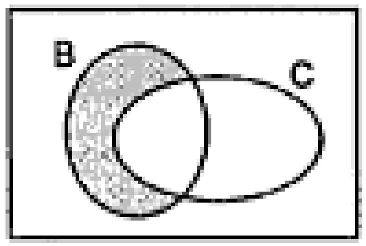
- Unification of two diagrams
  - Diagram D is obtainable from D_1 and D_2 by unification if and only if the following hold:
    - The set of labels of D is the union of the set of labels of D1 and the set of labels of D2
    - If a region r of either D1 or D2 is shaded, then there is a counterpart of it in D which is also shaded. Likewise, if any region r of D is shaded, then there is a counterpart of it in either D2 or D2 which is also shaded
    - If r is a region having an ⊗-sequence in either D1 or D2, then there is a counterpart of it in D which also as an ⊗-sequence. Similarly, if any region r of D has an ⊗-sequence, then there is a counterpart of it in either D1 or D2 which also has an ⊗-sequence

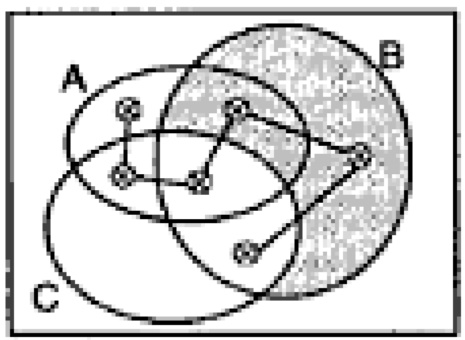
PROOFS IN DIAGRAMMATIC REASONING

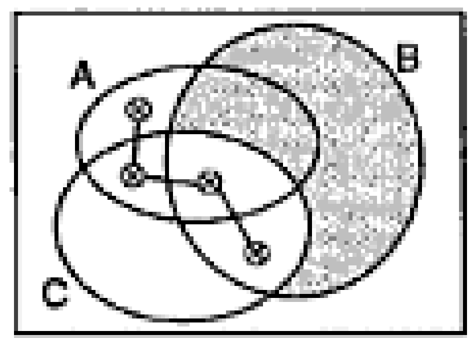
- A diagram D is “provable” from a set of diagrams Δ if and only if there is a sequence of diagrams $D_0,\ldots,D_n$ such that each diagram is a member of Δ or else is obtainable from earlier diagrams in the sequence by on of the rules of inference

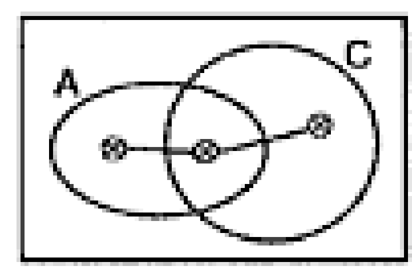
EXAMPLE PROOF











SOUNDNESS AND COMPLETENESS

SOUNDNESS

- Let Δ be a collection of diagrams, D a diagram. Then if $\Delta \vdash D$, $\Delta \models D$.
- Proven by Shin in 1991
- Since the proof involves unifying together all the diagrams in Δ, it can't be generalized to handle the infinite case.

COMPLETENESS

- Corollary of finite completeness

Corollary 4.9 (Completeness) *Let Δ be a set of diagrams, D a diagram. Then if $\Delta \models D$, $\Delta \vdash D$.*

Proof: Suppose that $\Delta \nvdash D$. Let $D_1, \ldots, D_n$ be the immediate subdiagrams of D. Then it must be the case that $\Delta \cup \{\neg D_i\}$ is consistent for some $1 \leq i \leq n$. Suppose otherwise. Then there would be a finite subset Δ_0 of Δ such that $\Delta_0 \cup \neg\{D_i\}$ is inconsistent for each $1 \leq i \leq n$. By Soundness, $\Delta_0 \cup \neg\{D_i\}$ would be unsatisfiable. So $\Delta_0 \models D_i$ for each i and hence $\Delta_0 \models D$. Thus $\Delta_0 \vdash D$ (by 4.2) and thus $\Delta \vdash D$, a contradiction. So, for some i, $\Delta \cup \{\neg D_i\}$ is consistent, and therefore satisfiable by the Model Existence Theorem. Let $M \models \Delta \cup \{\neg D_i\}$. Then M is a model of Δ but not of D_i. Thus M does not model D. Hence $\Delta \not\models D$, a contradiction. □

STRENGTHS OF DIAGRAMMATIC REASONING

- Visual representation of logic
- Multiple ways to represent data
- Able to reason from a visual representation of information

WEAKNESSES OF DIAGRAMMATIC REASONING

- Not good for modeling complex systems

STRENGTHS AND WEAKNESSES

ANY QUESTIONS?