Homogeneous Differential Equation
Corrected example

Begin with the homogeneous differential equation

$$(x^3 + x^2 y)\,dx + y^3\,dy. \tag{1}$$

Let $y = xv$, then $dy = x\,dv + v\,dx$. Substitution produces

$$\begin{aligned}
(x^3 + x^3 v)\,dx + x^3 v^3 (v\,dx + x\,dv) &= 0 \\
x^3(1 + v + v^4)\,dx + x^3 v\,dv &= 0 \\
(1 + v + v^4)dx + xdv &= 0 \quad \text{(Separable)} \\
\frac{dx}{x} + \frac{dv}{1 + v + v^4} &= 0
\end{aligned}$$

We use partial fractions. Since $1 + v + v^4 = \prod_{i=1}^{4}(v - \alpha_i)$, we have

$$\frac{1}{1 + v + v^4} = \sum_{i=1}^{4} \frac{A_i}{v - \alpha_i} = \sum_{i=1}^{4} \frac{A_i \prod_{j=1}^{3}(v - \alpha_{i+j})}{1 + v + v^4}.$$

Then $1 = A_i \prod_{j=1}^{3}(\alpha_i - \alpha_{i+j})$, and $A_i = \frac{1}{\prod(\alpha_i - \alpha_j)} = \frac{-\alpha_i}{3\alpha_i + 4}$.

Integrating, we have $\ln|x| + \sum_{i=1}^{4} A_i \ln|v - \alpha_i| = C_1$, so $x \prod (v - \alpha_i)^{A_i} = C$.

Since $v = y/x$, $x \prod (v - \alpha_i)^{A_i} = x \prod \left(\frac{y}{x} - \alpha_i\right)^{A_i}$. This simplifies (since $A_1 + A_2 + A_3 + A_4 = 0$) to $x \prod (y - \alpha_i x)^{A_i}$.

Therefore $x \prod (y - \alpha_i x)^{A_i} = C$ satisfies (1).

Technical note: Expanding the product $\prod(v - \alpha_i)$ produces $v^4 - \sum(\alpha_i)v^3 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4)v^2 - (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4)v + \alpha_1\alpha_2\alpha_3\alpha_4$. The coefficients are know as the symmetric polynomials. Note that each coefficient is homogeneous, and each homogeneous polynomial of degree d is the sum of all possible products of d of the roots. These coefficients are often abbreviated as $v^4 - s_1 v^3 + s_2 v^2 - s_3 v + s_4$. Thus, $s_1 = s_2 = 0$, $s_3 = -1$ and $s_4 = 1$.

Expanding $\prod_{j=1}^{3}(\alpha_i - \alpha_{i+j})$ gives

$$\alpha_i^3 - \alpha_i^2\Big(\sum_{j=1}^{3} \alpha_{i+j}\Big) + \alpha_i\alpha_{i+1}\alpha_{i+2} + \alpha_i\alpha_{i+1}\alpha_{i+3} + \alpha_i\alpha_{i+2}\alpha_{i+3} - \alpha_{i+1}\alpha_{i+2}\alpha_{i+3},$$

which is $\alpha_i^3 - \alpha_i^2(-\alpha_i) + s_3 - 2\alpha_{i+1}\alpha_{i+2}\alpha_{i+3}$, or $2\alpha_i^3 - 1 - \frac{2}{\alpha_i}$. Joining these with a common denominator produces $\frac{2\alpha_i^4 - \alpha_i - 2}{\alpha_i}$. Since α_i is a root of $v^4 + v + 1 = 0$, $\alpha_i^4 = -1 - \alpha_i$, so we may simplify to $\frac{-3\alpha_i - 4}{\alpha_i}$.

Hence, $A_i = \frac{1}{\prod_{j=1}^{3}(\alpha_i - \alpha_{i+j})} = \frac{\alpha_i}{-3\alpha_i - 4}$.

$v^4 + v + 1$ has no real roots, so the roots come in conjugate pairs, $\alpha, \overline{\alpha}$ and $\beta, \overline{\beta}$. Writing $\alpha = a + bi$, we note that the sum of the roots is 0, so $\beta = -a + ci$. This causes a great deal of simplification in the sums of combinations of the roots.