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# Thump, ring: the sound of a bouncing ball 

J I Katz<br>Department of Physics and McDonnell Center for the Space Sciences, Washington University, St Louis, MO 63130, USA<br>E-mail: katz@wuphys.wustl.edu

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#### Abstract

A basketball bounced on a stiff surface produces a characteristic loud thump, followed by a high-pitched ringing. Describing the ball as an inextensible but flexible membrane containing compressed air, I formulate an approximate theory of the generation of these sounds and predict their amplitudes and waveforms.


## 1. Introduction

A basketball bounced on a stiff surface, such as a thick concrete slab, emits a loud characteristic 'thump', followed by a high-pitched ringing. The sound is very different if the ball is bounced on a more resilient or softer surface. These characteristic sounds are not heard when a solid ball is bounced, and the bounce of a soft rubber ball of similar size produces the ringing but not the thump.

Basketballs (and several other types of ball, including volleyballs, (American) footballs and soccer (football outside the United States) balls) are inflated with air to an overpressure of about half atmospheric pressure. This overpressure gives them their stiffness and resilience. Beach balls and large balls meant for small children are also inflated, but to lower pressure. We describe the outer skin of the ball (which is made of various combinations of rubber, leather and nylon fibre, and various plastics in lower pressure balls) as an inextensible but perfectly flexible membrane. Inextensibility is a fair approximation, as shown by the fact that these balls expand only slightly upon inflation. Perfect flexibility is, at best, a rough approximation, but is necessary to describe the mechanics of bounce without resorting to a numerical elastodynamic treatment of the skin. In addition, we assume normal incidence and no rotation.

It is important to distinguish the mechanics of a ball consisting of a thin (and hence flexible) but nearly inextensible membrane pressurized by a gas contained within it from that of a solid elastic ball. This paper is concerned with the former case; the filling gas has no resistance to shear and the membrane has a great (nominally infinite) resistance to extension
(surface tension). Many other sports balls (baseballs, handballs, golf balls, etc) are solid objects whose mechanics are entirely different.

The phenomena described here are easily heard if one listens for them, and bouncing balls are everywhere. While there is a literature on the mechanics of sport, including the paths of balls, there have not been quantitative studies of the sounds emitted. The only published work applicable to the subject of this paper is [1], and that tests only the frequency of the ringing emission discussed in section 4. Unfortunately, experimental acoustics is difficult. The obstacles include the quantitative calibration of the sensitivity of microphones (amplitude standards are scarce), reverberation (to study a 40 Hz sound requires that there be no reflectors or scatterers within 4 m of the source, and measurement of the spectrum of a broad-band source centred at that frequency would require data extending to much lower frequencies, and hence an unobstructed space several times larger), noise (environmental noise outdoors and man-made noise indoors) and refraction by velocity fields and temperature gradients. My attempts to collect useful data with available equipment failed for these reasons. Hence this paper is limited to theoretical predictions. I hope that an experimentalist with better facilities can test these predictions quantitatively, and without the bias inevitable when someone is testing his own predictions.

## 2. Energetics

When an inflated ball strikes a stiff immovable flat surface, the portion of the ball pressing against the surface loses its spherical shape. The remainder of the ball remains spherical because inextensibility implies that latitude lines cannot stretch, and the volume is maximized if they retain their pre-impact spherical geometry rather than telescoping. Maximization of the volume also implies that the contacting portion of the ball, formerly a spherical cap, is pressed flat against the rigid surface (figure 1). This requires a complex pattern of crumpling, which we do not calculate. Because the skins of real balls have a finite thickness (about 3.5 mm for basketballs), the approximation of an infinitely flexible membrane is not accurate, but we make this approximation to permit a simple analytic treatment.

Flattening the end cap of a spherical ball, as shown in figure 1, requires a mechanical work

$$
\begin{equation*}
W=\int p \mathrm{~d} V \approx p \Delta V=\pi p\left(a x^{2}-x^{3} / 3\right) \approx \pi p a x^{2} \tag{1}
\end{equation*}
$$

where $x>0$ is the depth of flattening of the ball (the distance its centre has travelled normal to the surface after first contact) and $a$ is its unflattened radius. Taking $x \ll a$, we have approximated $p$, the excess pressure inside the ball over ambient pressure ( $p$ is also known as the gauge pressure), as a constant.

The mechanical work has the form of a simple harmonic oscillator potential

$$
\begin{equation*}
W=\frac{1}{2} k x^{2} \tag{2}
\end{equation*}
$$

with spring constant

$$
\begin{equation*}
k=2 \pi p a . \tag{3}
\end{equation*}
$$

As a result, the motion of the ball when it is in contact with the surface is sinusoidal

$$
\begin{equation*}
x=x_{0} \sin \omega t \tag{4}
\end{equation*}
$$

for $0 \leqslant t \leqslant \pi / \omega$, with frequency

$$
\begin{equation*}
\omega=\left(\frac{2 \pi p a}{M}\right)^{1 / 2} \tag{5}
\end{equation*}
$$



Figure 1. An inflated (pressurized) spherical ball made of an inextensible but flexible skin recoiling from a stiff flat surface. The ball retains its spherical shape except where in contact with the surface. The skin of the flattened spherical cap crumples to accommodate its reduced surface area.
where the mass $M$ includes the mass of the skin of the ball, its contained air and the induced mass of the surrounding airflow. To an excellent approximation for most inflated balls used in sports, $M$ is given by the mass of the solid (polymer) skin.

The ball remains in contact with the surface for a half-cycle of this oscillation, independent of its velocity $v_{0}=x_{0} \omega$ at impact. If dropped from a height $h$, then $v_{0}=\sqrt{2 g h}$, ignoring air drag during its fall (for $h=1 \mathrm{~m}$ air drag introduces an error $\mathcal{O}(10 \%)$ ).

## 3. Monopole emission-the 'thump'

The changing volume of the ball produces a monopole source [2] of sound described by the pressure field

$$
\begin{equation*}
p_{\mathrm{rad}}(r, t)=\frac{\rho}{2 \pi r} Q^{\prime}(t-r / c), \tag{6}
\end{equation*}
$$

where $\rho$ is the density and $c$ is the sound speed of the surrounding fluid, the volume flow rate

$$
\begin{equation*}
Q(t)=\frac{\mathrm{d} V}{\mathrm{~d} t} \tag{7}
\end{equation*}
$$

and we have multiplied the standard result by a factor of 2 to allow for the fact that at the surface of an infinite rigid slab sound is radiated into only $2 \pi$ sterad. For a sphere with a small flattened endcap $(x \ll a)$

$$
\begin{equation*}
V \approx V_{0}-\pi a x^{2} \tag{8}
\end{equation*}
$$

Using (4), (7) and (8), we obtain

$$
\begin{equation*}
p_{\mathrm{rad}}(r, t) \approx-\frac{\rho a v_{0}^{2}}{r} \cos 2 \omega(t-r / c) \tag{9}
\end{equation*}
$$

for $0 \leqslant t-r / c \leqslant \pi / \omega$, and zero otherwise.


Figure 2. The acoustic signal as a function of time (time normalized to the period $2 \pi / \omega$ ). The actual time dependence is one cycle of simple harmonic motion of angular frequency $2 \omega$.

Standard basketballs have $M \approx 0.6 \mathrm{~kg}(600 \mathrm{~g}), p \approx 5.5 \times 10^{4} \mathrm{~Pa}\left(5.5 \times 10^{5}{\text { dynes } \mathrm{cm}^{-2}}\right.$, $8 \mathrm{psi})$, and an internal radius $a \approx 0.114 \mathrm{~m}(11.4 \mathrm{~cm})$, yielding $\omega \approx 256 \mathrm{~s}^{-1}$. The acoustic pulse described by (9) varies sinusoidally at an angular frequency $2 \omega$, corresponding to 82 Hz , as shown in figure 2. The origin of the double frequency is the quadratic dependence of $V$ on $x$ (8).

The power spectrum of $p_{\text {rad }}$ is shown in figure 3 . The peak at a frequency $\approx \omega$ results from the peak in $p_{\mathrm{rad}}(t)$ at $1 / 4$ of a cycle of period $2 \pi / \omega$, as shown in figure 2 . The zeros in the power spectrum at higher integer multiples of $\omega$ result from the orthogonality relations of trigonometric functions, and its peaks occur between them. Because the acoustic amplitude is zero outside a short interval of length $\pi / \omega$ (half the simple harmonic period of the surface rebound) its spectrum is broad, giving a 'thump' rather than narrow-band ringing. Attempting to resolve the spectrum on shorter time scales would not be meaningful [3].

Taking $v_{0}=4.43 \mathrm{~m} \mathrm{~s}^{-1}(h=1 \mathrm{~m})$ and $\rho=1.19 \mathrm{~kg} \mathrm{~m}^{-3}\left(1.19 \times 10^{-3} \mathrm{gm} \mathrm{cm}^{-3}\right)$ (dry air at $20^{\circ} \mathrm{C}$ and standard pressure) yields a peak pressure amplitude of $0.89 \mathrm{~Pa}\left(8.9\right.$ dynes $\left.\mathrm{cm}^{-2}\right)$, or 93 dB (referred to the standard 0 dB level of $2 \times 10^{-5} \mathrm{~Pa}\left(2 \times 10^{-4}\right.$ dyne $\left.\mathrm{cm}^{-2}\right)$ ) at a range $r=3 \mathrm{~m}$. This explains the surprising loudness of the 'thump'.


Figure 3. Power spectrum of the 'thump'. The frequency is given in multiples of that of equation (5) $(41 \mathrm{~Hz})$ and the power density is normalized to its peak value.

## 4. Dipole emission-the 'ring'

In addition to the 'thump', a high-pitched ringing sound is also heard. When a softer ball inflated to low pressure, such as a beach ball or a small child's toy, is used, the 'thump' is less audible because $\omega$ is very low, and the ringing is more striking. The frequency is that of the lowest eigenmode of the oscillation of the air inside the ball.

On long time scales $\mathcal{O}(1 / \omega)$ it is possible to think of the force of contact with the slab as transferring momentum to the ball as a whole. On the much shorter time scales characteristic of the ball's internal eigenmodes the internal response of the contained air must be considered. The air acts as a nearly massless spring between the fixed slab and the more massive ball. However, because the air has some mass and the frequencies of the internal modes are finite, though large, some energy is coupled to them.

In the inextensible approximation, the skin of the ball may be considered to be a rigid spherical shell after contact with the flat surface is broken. The governing equation for sound
waves is

$$
\begin{equation*}
\nabla^{2} p+\frac{\omega^{2}}{c^{2}} p=0, \tag{10}
\end{equation*}
$$

where $\omega$ is now the frequency of the acoustic wave. Azimuthally symmetric modes are given (inside the ball) by

$$
\begin{equation*}
p(r, \theta)=P_{\ell}(\cos \theta) j_{\ell}(k r) \tag{11}
\end{equation*}
$$

where the wave vector $k \equiv \omega / c . \quad P_{\ell}$ is the Legendre polynomial of order $\ell$ and $j_{\ell}$ is the spherical Bessel function of that order.

The value of $k$ is determined by the boundary condition $d j_{\ell}(k r) / \mathrm{d} r=0$ at $r=a$ because at the outer surface $0=v_{r}=\frac{\partial j_{\ell}(k r)}{\partial r} P_{\ell}(\cos \theta) /(\mathrm{i} \omega \rho)$. The boundary condition on the parallel component of velocity is not applicable outside a boundary layer of thickness $\mathcal{O}\left(\sqrt{v / \omega_{1}}\right) \sim 0.05 \mathrm{~mm}$, where $v$ is the kinematic viscosity.

The lowest frequency mode is found for $\ell=1$, corresponding to a mode in which all the air moves in the same direction at any one time, and the diameter is approximately a half-wavelength $(k a=2.082)$. The $\ell=0$ mode has a velocity node at the origin as well as at the surface, so the diameter is approximately a full wavelength ( $k a=4.494$ ). Its frequency is roughly twice as high as that of the $\ell=1$ mode. Hence the lowest mode frequency is

$$
\begin{equation*}
\omega_{1}=\frac{2.082 c}{a} \tag{12}
\end{equation*}
$$

At a temperature of $20^{\circ} \mathrm{C}\left(c=3.43 \times 10^{2} \mathrm{~m} \mathrm{~s}^{-1}\right)$ we find $\omega_{1}=6.26 \times 10^{3} \mathrm{~s}^{-1}(997 \mathrm{~Hz})$, in agreement with [1].

It is possible to estimate the amplitude of the $\ell=1$ mode. It is excited, in the inextensible membrane approximation, by the force the rigid surface exerts on the air when the ball is in contact with the surface. In this approximation, the force is transmitted through the flattened portion of the skin to the contained air. A slowly varying (compared to the frequency $\omega$ ) pressure gradient in that air accelerates the skin (which contains nearly all the mass).

The momentum imparted to the oscillating air then equals the Fourier amplitude of the applied force $F(t)$ at the mode frequency $\omega_{1}$. The resulting momentum is

$$
\begin{equation*}
P_{\omega_{1}}=\int_{0}^{\pi / \omega} F(t) \operatorname{expi} \omega_{1} t \mathrm{~d} t=\int_{0}^{\pi / \omega} p A(t) \operatorname{expi} \omega_{1} t \mathrm{~d} t, \tag{13}
\end{equation*}
$$

where $A(t)$ is the area of contact. Using $A(t)=2 \pi a x$ and equation (4) we find

$$
\begin{equation*}
P_{\omega_{1}}=2 \pi a p x_{0} \int_{0}^{\pi / \omega} \sin \omega t \operatorname{expi} \omega_{1} t \mathrm{~d} t \tag{14}
\end{equation*}
$$

The integral is complex, but we are only interested in its modulus

$$
\begin{equation*}
\left|P_{\omega_{1}}\right|=2 \pi a p x_{0} \frac{\omega}{\omega_{1}^{2}-\omega^{2}}\left[2\left(1+\cos \pi \omega_{1} / \omega\right)\right]^{1 / 2} . \tag{15}
\end{equation*}
$$

Because $\omega_{1} \gg \omega$ the factor in the square bracket is unknowable, but its root mean squared value may be found by averaging over a full cycle of the cosine. The resultant root mean squared momentum

$$
\begin{equation*}
\left\langle P_{\omega_{1}}\right\rangle=2 \sqrt{2} \pi a p x_{0} \frac{\omega}{\omega_{1}^{2}-\omega^{2}} \tag{16}
\end{equation*}
$$

It is possible to integrate the velocity field over the interior of the ball to find the pressure and velocity amplitude corresponding to $P_{\omega_{1}}$. Writing $p_{1}(r, \theta)=a_{1} P_{1}(\cos \theta) j_{1}(k r)$ we find

$$
\begin{equation*}
a_{1}=\frac{\operatorname{pax}_{0}}{c^{2}} \frac{\omega^{4}}{\omega_{1}^{2}-\omega^{2}} \frac{\sqrt{2\left(1+\cos \pi \omega_{1} / \omega\right)}}{3.463} . \tag{17}
\end{equation*}
$$

We are more interested in the acoustic radiation by this mode, which can be obtained directly from $P_{\omega_{1}}$. The oscillation of the air within the ball produces an opposite oscillation of the ball, with velocity amplitude $U_{1}=P_{\omega_{1}} / M$. The dipole approximation is not valid because $k a=2.082$ rather than $k a \ll 1$. However, Morse [2] tabulates numerical solutions for general $k a$. The resulting radiation field

$$
\begin{equation*}
p_{\mathrm{rad}}(r, t)=\frac{\rho c U_{1}}{k r D_{1}} \cos \theta \cos \omega(t-r / c) \tag{18}
\end{equation*}
$$

where for $k a=2.082$ the factor $D_{1}=0.531$. For our previous parameters $x_{0}=v_{0} / \omega=$ 17 mm , and we find $\left\langle P_{\omega_{1}}\right\rangle=0.0075 \mathrm{~kg} \mathrm{~m} \mathrm{~s}^{-1}\left(750 \mathrm{gm} \mathrm{cm} \mathrm{s}^{-1}\right), U_{1}=0.0125 \mathrm{~m} \mathrm{~s}^{-1}$ $\left(1.25 \mathrm{~cm} \mathrm{~s}^{-1}\right)$ and the amplitude of $p_{\mathrm{rad}}$ at $r=3 \mathrm{~m}$ and $\theta=0$ is $0.187 \mathrm{~Pa}\left(1.87\right.$ dynes $\left.\mathrm{cm}^{-2}\right)$, or 79 dB . This pressure amplitude is 14 dB below that of the low frequency 'thump' but is readily audible.

In the approximation of an inextensible membrane the $\ell=0$ mode is not excited, to lowest order in the shape change during surface contact, because its velocity distribution does not change the volume. Even were it excited, in this approximation it would not radiate because, once surface contact is broken, the volume of the ball is constant and there is no monopole radiation; higher multipoles are zero for this mode.

## 5. The inextensible approximation

The validity of the inextensible membrane approximation may be quantified by comparing the frequency $\omega$ which describes the bounce to the frequency $\omega_{e l}$ of the 'breathing' mode in which the skin oscillates, spherically symmetrically, about its equilibrium radius. The inextensible approximation corresponds to the limit $\omega_{e l} / \omega \rightarrow \infty$.

By force balance we see that inflating the ball to an overpressure $P$ increases its radius by an amount

$$
\begin{equation*}
\delta a=\frac{P a^{2}}{h E} \tag{19}
\end{equation*}
$$

where $h$ is the skin's thickness and $E$ its Young's modulus. Young's moduli of rubber are found [4] over a wide range, roughly $10^{6}-10^{8} \mathrm{~Pa}\left(10^{7}-10^{9}\right.$ dynes $\left.\mathrm{cm}^{-2}\right)$, while that of leather is typically [5] about $5 \times 10^{7} \mathrm{~Pa}\left(5 \times 10^{8}\right.$ dynes $\left.\mathrm{cm}^{-2}\right)$. Basketballs generally are stiffened with wound nylon filament with a modulus[4] of about $3 \times 10^{9} \mathrm{~Pa}\left(3 \times 10^{10}\right.$ dynes $\left.\mathrm{cm}^{-2}\right)$. If $P$ is measured with a pressure gauge, then measuring $\delta a$ may be the most convenient means of determining $E h$.

The breathing mode oscillation is described by a potential energy

$$
\begin{equation*}
U=-P \mathrm{~d} V+2 \times 4 \pi a^{2} h \frac{u \sigma}{2} \tag{20}
\end{equation*}
$$

where $u$ is the strain and $\sigma$ is the stress, and the first factor of 2 comes from the fact that the skin is stretched along two orthogonal axes. For small strains $u \approx \delta a / a, \sigma \approx E \delta a / a$ and

$$
\begin{equation*}
U \approx-4 \pi a^{2} P \delta a+4 \pi(\delta a)^{2} h E \tag{21}
\end{equation*}
$$

yielding an effective spring constant $k_{\text {breathe }}=\partial^{2} U / \partial \delta a^{2}=8 \pi h E$. Then, using $M=$ $4 \pi a^{2} h \rho_{\text {skin }}$,

$$
\begin{equation*}
\omega_{e l}=\left(\frac{2 E}{\rho_{\text {skin }} a^{2}}\right)^{1 / 2} \tag{22}
\end{equation*}
$$

The static inflation pressure drops out because the elastic response is assumed to be linear.

The ratio of the breathing mode frequency to the bounce frequency is

$$
\begin{equation*}
\frac{\omega_{e l}}{\omega}=\left(\frac{4 E h}{P a}\right)^{1 / 2}=\left(\frac{4 a}{\delta a}\right)^{1 / 2} . \tag{23}
\end{equation*}
$$

For an inextensible membrane $E \rightarrow \infty$ and $\omega_{e l} / \omega \rightarrow \infty$. The ratio $a / \delta a$ is readily measured as the ball is inflated. A rough measurement on a Spalding 'NBA Indoor/Outdoor' basketball yielded $\delta a / a=0.009$ for $P=5.5 \times 10^{4} \mathrm{~Pa}\left(5.5 \times 10^{5}\right.$ dynes $\left.\mathrm{cm}^{-2}, 8 \mathrm{psi}\right)$. The breathing mode frequency $\omega_{e l} / 2 \pi$ is about 900 Hz (accidentally close to the ringing frequency), $E h=7 \times$ $10^{5} \mathrm{~J} \mathrm{~m}^{-2}\left(7 \times 10^{8} \mathrm{erg} \mathrm{cm}^{-2}\right)$ and $E=2 \times 10^{8} \mathrm{~Pa}\left(2 \times 10^{9} \mathrm{erg} \mathrm{cm}^{-3}\right)$. The stiffness is largely provided by the nylon filaments.

The ratio $\omega_{e l} / \omega=21$ is substantially, but not enormously, larger than unity, so inextensibility should be a reasonable approximation. The infinite flexibility approximation may not be accurate, but is not so readily quantified, and the actual dynamics at impact involves such complications as friction with the surface.

The most important consequence of the breakdown of the inextensibility approximation is that impact excites the breathing mode. After the ball breaks contact with the surface it continues to radiate. The rubbery materials of which basketballs are made have large loss coefficients, so this oscillation and radiation damp in about a single cycle.

In the opposite (extensible membrane, incompressible filling fluid) approximation, the volume is not changed at impact and there is neither 'thump' nor 'ring'. This limit is applicable to a water-filled latex balloon, for which there are only very low frequency volume-conserving elastic oscillations analogous to the quadrupole (and higher multipole) oscillations of liquid drops. An air-filled latex balloon is an intermediate case, and some 'thump' is heard. There is no ringing because the low impedance membrane permits internal motions to couple to the surrounding air; there are no internal standing wave modes.

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