A simple, sensitive circuit to measure Boltzmann's constant from Johnson's noise

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Abstract
A simple, easy to build electronic circuit consisting of an amplifying chain followed by a state-variable bandpass filter has been used to measure the thermal noise produced by a resistor. Using Johnson's theory for thermal noise, Boltzmann's constant could be derived with an accuracy of 1.4% of its accepted value. This simple instrument can be very useful for modern physics laboratories due to its low cost and use of standard instrumentation.

1. Introduction

Thermal noise is generated by any material at a temperature above absolute zero. Noise voltages may be specified by their amplitude distribution over their spectral bandwidth. One of the most common kinds of noise is the band-limited, white, Gaussian noise, which is a signal with equal power per unit frequency (Hz) over a certain band of frequencies and with a Gaussian distribution of amplitudes. This kind of noise is generated by a resistor (Johnson noise), and it affects sensitive measurements of all kinds. The link between the macroscopic parameters accessible to measurement and the microscopic phenomena occuring in the resistor was provided by Johnson in 1928 [1]. Briefly, Johnson showed that the average of the square of the voltage measured across the terminals of a conductor is proportional to its resistance and temperature, and that it does not depend on the material of which the conductor is made nor on any other physical or chemical properties. Nyquist's theory, published in the same year [2], provided a rigorous theoretical explanation based on the principles of statistical thermodynamics.

There already exist several experimental layouts to measure Boltzmann's constant from the measurement of the thermal noise of a resistor [3–5]. However, these setups use rather expensive instrumentation, and some of them are intended for precise thermometry. Ericson [6] has reported the design of a simple system to measure Boltzmann's constant, using a high frequency amplifier (subject to more noise problems) and a passive bandpass filter. No reference is made in this paper about the particular value obtained for Boltzmann's constant.
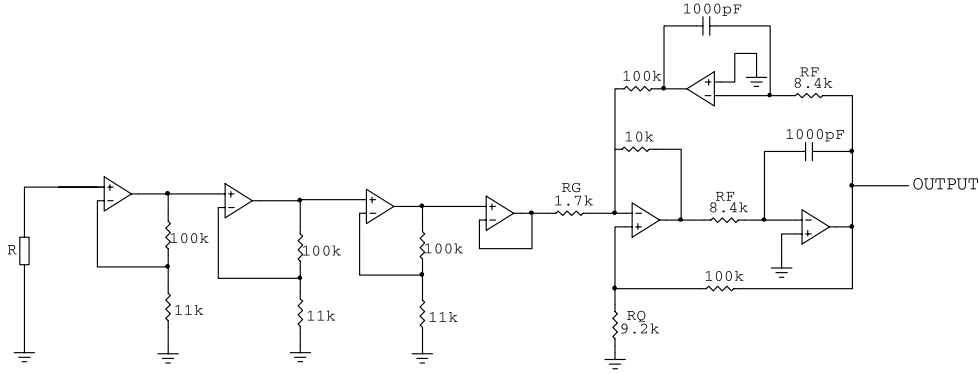
Figure 1. Diagram of the electronic circuit for the measurement of Johnson's noise, produced by the resistor R. All operational amplifiers are LT1024.

Rather, it is only claimed that it differs from the accurate value only by a few per cent. This paper reports the design of a simple electronic circuit that is capable of providing useful signals out of which Boltzmann's constant can be derived with an uncertainty of 1.4%, a value which is considered good from the point of view of an experiment for undergraduates.

Even though electrons are in motion in a resistor, there is no net flow of current across its terminals, and hence the average voltage is zero:

$$\langle V(t) \rangle = 0. \tag{1}$$

However, thermal fluctuations do generate a finite voltage as a function of time, that is, $V(t) \neq 0$, and hence the variance of $V(t)$ is finite, that is

$$\sigma_V^2 = \langle (V - \langle V \rangle)^2 \rangle = \langle V^2 \rangle - \langle V \rangle^2 = \langle V^2 \rangle \neq 0, \tag{2}$$

and this quantity, accessible to measurement, is termed the thermal Johnson voltage.

If a resistor R is connected across the input terminals of an amplifier of gain $g(\omega)$, where ω is the angular frequency, and it is tuned so as to pass angular frequencies over a predetermined range, then the fluctuating current $I(t)$ would give rise to a variation of the voltage $V(t)$ across the terminals of the amplifier to which the resistor is connected, as is shown in figure 1. The elementary contribution dV^2 to the total mean square voltage at the output of the bandpass filter in a differential frequency interval is [7]

$$dV^2 = [g(\omega)]^2 \, dV'^2, \tag{3}$$

where V'^2 is the voltage measured across the resistor (without any amplification), and $g(\omega) = V_{\text{out}}(\omega)/V_{\text{in}}(\omega)$ is the overall gain of the amplifier and bandpass filter combination as a function of frequency. If $V(t)$ is expressed in terms of Fourier components, then one can write [7]

$$\langle V^2 \rangle = \int_0^\infty [g(\omega)]^2 f(\omega) \, d\omega, \tag{4}$$

where $f(\omega)$ is the spectral density of $V(t)$. In deriving $f(\omega)$, Nyquist considered that the resistor could be viewed as a simple, one-dimensional case of blackbody radiation, and hence this gives [3]

$$f(\omega) = \frac{2}{\pi} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} R, \tag{5}$$

where $\hbar = h/2\pi$ (h is Planck's constant) and $\beta = 1/kT$ (where k is Boltzmann's constant and T is the temperature). Thus we have from equation (4)

$$\langle V^2 \rangle = \frac{2R}{\pi} \int_0^\infty [g(\omega)]^2 \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \, d\omega. \tag{6}$$

Since at room temperature $kT \gg \hbar\omega$, and with $\omega = 2\pi\nu$ where ν is the natural frequency, one obtains

$$\langle V^2 \rangle = 4kTR \int_0^\infty [g(\nu)]^2 \, d\nu. \tag{7}$$

Hence, Johnson's voltage at room temperatures is given by [2]

$$\langle V^2 \rangle = 4kTRS, \tag{8}$$

where S is the gain factor, defined as

$$S \equiv \int_0^\infty g^2(\nu) \, d\nu. \tag{9}$$

The S factor is due to the need of amplifying the noise signal generated by the resistor, since it itself is very small, and hence inaccessible to direct measurement. We shall use equation (8) to derive k from the measurement of $\langle V^2 \rangle$.

2. Measurement of Johnson's noise

A diagram of the measurement setup is shown in figure 1. A resistor of value R is closely connected to the input terminals of a low-noise amplifier of voltage gain 1000; the amplified signal is then applied to an active bandpass filter with a voltage gain 10 over its bandpass, and measured from a simple digital oscilloscope of 40 MHz bandwidth. The amplifier module was built out of a cascade of three non-inverting amplifiers, each of gain 10; the very low-noise operational amplifier LT1024 was used throughout. The reason underlying the choice of three amplifying stages stems on the need to keep a reasonable bandwidth, while not generating a substantial noise. The overall bandwidth of the amplifying system is DC–200 kHz, which embraces perfectly the bandwidth of the bandpass filter. All operational amplifiers are biased by a dual, well-regulated, very low noise supply (± 15 V). A couple of 10 μF, tantalum capacitors should be connected to the ± 15V feeding lines of the printed circuit board.

For the bandpass filter, we have selected a state-variable active filter as described in [8]. This filter was designed for a bandpass gain of $G=10$, with low and high cutoff frequencies of $\nu_l = 1$ kHz and $\nu_h = 11$ kHz, respectively, both defined to occur after a 3 dB decay from the bandpass gain.

The quality factor of the circuit is defined by

$$Q = \frac{\nu_c}{\nu_h - \nu_l}. \tag{10}$$

Choosing a mid-band pass frequency of $\nu_c = 6$ kHz, we obtain $Q = 0.6$, for a bandwidth of $\nu_h - \nu_l = 10$ kHz. R_F is used for setting the centre frequency, while R_Q and R_G define both Q and the band-centre gain [8]. Thus, one can build a tunable-frequency, constant-Q filter by simply adjusting R_F. The resistors are calculated from the following relations [8]:

$$R_F = \frac{5.03 \times 10^7}{\nu_c} \Omega, \tag{11}$$

$$R_Q = \frac{1.0 \times 10^5}{(3.48Q + G - 1)} \Omega, \tag{12}$$

$$R_G = \frac{3.16 \times 10^4 Q}{G} \Omega. \tag{13}$$

The S factor was obtained by applying a small sinewave signal to the combination of the amplifier and bandpass filter. Care was taken to apply a signal small enough so as to ensure
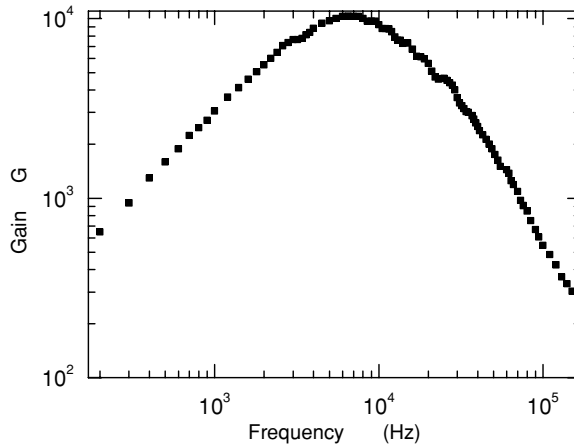
Figure 2. The measured, overall gain of the amplifier + bandpass filter combination as a function of frequency.

that the amplifier was operating in its linear region over the whole bandwidth of interest (DC–200 kHz), so that signal distortion was avoided. A simple voltage divider can be placed between the signal generator output and the amplifier input to accomplish this. A simple oscilloscope with a 20 MHz bandwidth or more may be used.

Several sources of noise in the environment may affect the measurement of Johnson's signals. Some of these spurious signals may arise from electrical equipment such as motors and power switches, and also from radiofrequency sources. Thus, it is essential to shield the whole electronic circuit from any of these sources. This was accomplished by placing the whole circuit inside a metallic enclosure, which can simply be a cardboard box wrapped with aluminium foil. It is essential to connect this box to a good ground. All connecting cables should also be wrapped with aluminium foil, and the choice of a single grounding point for all cables is essential to avoid induction loops.

The ratio between the output and input amplitudes is the gain $g(\nu)$, and its curve is shown in figure 2. This gain curve was measured by applying a small signal ($\sim$1 mV) from a sinewave generator.

The S term given by equation (9) was evaluated from the numerical integration of the gain curve given in figure 2, rendering a value of $S = 1.66 \times 10^{12}$ Hz.

3. Measurement of Boltzmann's constant

The $\langle V^2 \rangle$ values of the above circuit were measured for a number of resistors with values between 4.6 kΩ and 11.8 kΩ, as shown in figure 3. Theoretically, a straight line of slope $4kTS$ should be expected. The solid line connecting the experimental points represents the linear fit to $\langle V^2 \rangle$ with a slope

$$\frac{\langle V^2 \rangle}{R} = (2.78 \pm 0.06) \times 10^{-8} \ V^2 \ \Omega^{-1}, \tag{14}$$

and an intercept to the ordinate axis of value $4.0 \times 10^{-4} V^2$. It is important to note that this value is Johnson's noise due to the electronic circuit only when its input terminals are short circuited. Thus we confirm that equation (8) is valid independently of the nature of the resistor, since the intercept falls within the linear fit.
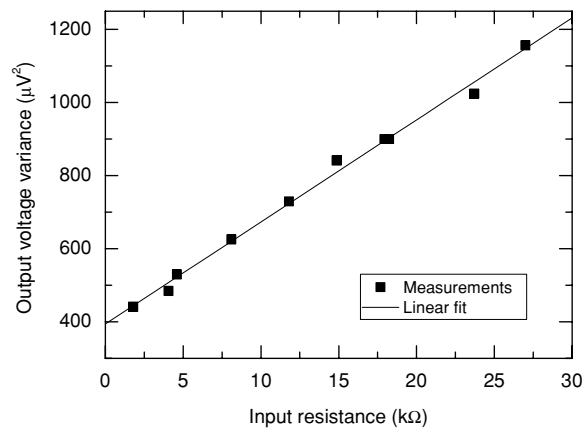
Figure 3. The variance of the output voltage of the whole measuring circuit as a function of the resistor connected across its input terminals.

Considering the temperature at which the measurements were performed ($T = 298 \pm 0.05$ K) and using equations (8) and (14), the value of Boltzmann's constant results in

$$k = \frac{\langle V^2 \rangle / R}{4TS} = (1.40 \pm 0.03) \times 10^{-23} \text{ J K}^{-1}. \tag{15}$$

This value is only 1.4% higher than the accepted value of $k = 1.38066 \times 10^{-23}$ J K^{-1}.

4. Conclusions

We have presented a simple, inexpensive circuit to determine Boltzmann's constant from the measurement of Johnson's noise from a resistor. The setup is easy to build, requires non-expensive instrumentation for the measurement of the noise voltage, and will provide the student with a direct, sound insight of thermal noise, its measurement and the derivation of Boltzmann's constant.

Acknowledgments

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