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Nonlinear dynamics of a magnetically driven Duffing-type spring–magnet oscillator in the static magnetic field of a coil

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Abstract
We study the nonlinear oscillations of a forced and weakly dissipative spring–magnet system moving in the magnetic fields of two fixed coaxial, hollow induction coils. As the first coil is excited with a dc current, both a linear and a cubic magnet-position dependent force appear on the magnet–spring system. The second coil, located below the first, excited with an ac current, provides the oscillating magnetic driving force on the system. From the magnet–coil interactions, we obtain, analytically, the nonlinear motion equation of the system, found to be a forced and damped cubic Duffing oscillator moving in a quartic potential. The relative strengths of the coefficients of the motion equation can be easily set by varying the coils’ dc and ac currents. We demonstrate, theoretically and experimentally, the nonlinear behaviour of this oscillator, including its oscillation modes and nonlinear resonances, the fold-over effect, the hysteresis and amplitude jumps, and its chaotic behaviour. It is an oscillating system suitable for teaching an advanced experiment in nonlinear dynamics both at senior undergraduate and graduate levels.

(Some figures may appear in colour only in the online journal)

1. Introduction

We present a theoretical model developed for a simple and weakly dissipative, nonlinear, magneto-mechanical oscillator, and the experimental results that validate the model. This oscillator consists of a short, cylindrical ferrite magnet hanging from a soft steel spring that oscillates vertically in the induction field of a short hollow cylindrical coil. It is excited with a constant current, and externally driven by the magnetic field of another, much larger coil excited with an ac current, as shown in figures 1 and 2. Forced oscillating systems like this one continue to engender great interest because of their special attributes and
the number of striking effects they can show. These include fold-over or tilted resonance curves and the striking modal hysteresis cycles associated with such resonances [1, 2], symmetry-breaking pitchfork bifurcations, sub-harmonics, attractors and the transition to chaos [3]. Nonlinear oscillator models are also used in many fields of to represent either extraordinary or little understood dynamic effects of useful systems, as for instance in nonlinear and quantum optics, fluid mechanics, biology and physiology, and even in economics [4, 5]. Several nonlinear oscillators have been presented in books and physics journals in the last four decades [1, 2, 6–11], most frequently the Duffing oscillator [2, 6, 7, 11, 12], modelled using the well-known differential equation of the same name. It usually takes the form of a weakly dissipative system subjected to strong external driving. Interestingly enough, this model can be applied to investigate important natural phenomena, e.g., off-shore structures subjected to the thrust of ocean waves, structural dynamics and resonance in trees, ecological systems subjected to yearly variation in temperature and solar light influx, and the hormonal system subject to the day-and-night rhythm in activity, food intake and light intensity. The applications of the Duffing oscillator model also include mechanical and magneto-mechanical systems with many components, and at times difficult-to-assemble and difficult-to-drive oscillators (e.g. figure 14.6 in [2]). Here we offer an attractive alternative, since the oscillator presented shows a number of advantages over its precursors, e.g., its simplicity and low cost, and the possibility of assembling it with pieces of equipment and ordinary measuring instruments available in any laboratory. It also offers the experimenter the possibility of directly observing large amplitude nonlinear oscillations, and the drastic nonlinear changes in system oscillation modes during hysteresis, all at low-frequency oscillations, easily observable with the naked eye.

When the small coil in figure 1 is excited with a constant current, it generates a variable restoring force on the oscillating spring–magnet system and we show that it can be represented
Figure 2. Actual experimental setup. The small coil and the ferrite magnet hanging from the spring can be seen at the centre and the large diameter coil is at the bottom of the picture. The oscilloscope-screen inset shows the ac driving current (upper trace) and the monitored chaotic large oscillations of the magnet when driven by the magnetic fields of both coils.

as a sum of terms of the form $-\alpha z - \beta z^3$, $z$ being the magnet vertical position along the main symmetry axis of the coil. The second and larger coil in figure 1 provides a harmonically varying driving force $f \cos(\gamma t)$ on the moving magnet when excited by an oscillating current of adjustable angular frequency $\gamma$. The truly nonlinear motion of this spring–magnet system can be observed easily with the naked eye, and the parameters of its motion can easily be set within a convenient range by simply tuning the electrical currents through the two coils. In a previous work [11], we studied the nonlinear oscillations of the same system when moving in the constant magnetic field of the small coil but in the absence of the large coil in figure 1, i.e. with no oscillating driving force applied, and showed that the magnet potential energy function is a quartic one, with two bottom sub-wells for given values of the dc current in the coil, and demonstrated how the magnet can jump from one sub-well to the other.

Section 2 of this paper is devoted to a detailed description of the oscillating system, and section 3 to the development of its physical model. In section 4 we describe the experiments we have performed to demonstrate the nonlinear behaviour of our oscillating system and to a comparison of the experimental results and the mechanical motion predicted by the model. Section 5 is devoted to a discussion and our final comments. We are confident that the experiments and the analytical model developed below, are an excellent example of the
study of nonlinear dynamics and classical electrodynamics for both advanced undergraduate and graduate physics laboratories.

2. Experimental setup

Figure 1 depicts a 2.20 cm diameter, 2.54 cm long cylindrical ferrite magnet of mass $M = 0.065$ kg, a magnetic dipole moment of $m = 2.6$ A·m$^2$, hanging from a soft stainless-steel spring of natural elastic, constant $k = 3.17$ N·m$^{-1}$, and 25 mm relaxed length. The magnet can oscillate along the vertical symmetry axis $z$ of the magnetic field of a fixed and hand-made hollow coil of $N = 20$ turns of 1.15 mm diameter copper wire. The mean radius of this coil is $a = 19.5$ mm, its length is $L = 23$ mm, and it is fed by a low voltage power supply, the typical current through it being $i_0 = 0.5$ A. The magnet mid-plane is initially made to coincide with the coil mid-plane, thus defining the zero of the magnet’s vertical position. One can monitor the coincidence of these two mid-planes by simply setting by hand the relative vertical position of the magnet and coil, and then exciting the coil with a small current; if there is coincidence of the two mid-planes within about $1/4$ of a millimetre, the magnet remains at rest, otherwise the magnet will begin oscillating appreciably. A second and larger diameter coaxial induction coil lies below the first coil. It is deliberately located just at the previously calculated vertical distance, about 16 cm below, where the radial field component of the magnet reaches its maximum at the turns of this coil. The coil was made with 270 turns of 1.6 mm copper wire, its total resistance barely reaching 2.3 Ω. Its mean radius is 16 cm, and its height 2 cm. The excitation current in this large coil has mean amplitude of 0.25 A, with a low temporal frequency in the range 0.5 to 1.5 Hz, provided by a signal generator followed by a home made current amplifier.

A plastic strip of varying optical transparency (home made by copying a linear greyscale on an acetate sheet using a digital printer) is placed between the lower end of the spring and the magnet (not drawn in figure 1. This acetate strip is used to modulate a collimated horizontal beam of light (from a white LED), in an approximate linear way as the magnet moves up and down [11]. By measuring the transmitted light beam intensity with a phototransistor, the varying vertical coordinate of the magnet can be monitored and conveniently stored in a digital oscilloscope. Figure 2 is an actual picture of the experiment setup: the small induction coil with the black ferrite magnet partially inserted is seen at the centre of the picture. The light filtering acetate strip can be seen between the vertical spring and the ferrite magnet.

Before each experiment, the oscillating parameters of this oscillating system are initially set or calibrated by separately varying the two coil currents, e.g., the natural frequency and the small attenuation coefficient of the spring–magnet free oscillations can be measured initially by turning off the two coils and monitoring the magnet oscillations with the digital oscilloscope. The natural weak damping of the oscillating system can be assessed by observing its long-lasting free oscillation in the oscilloscope. Then, by exciting the first coil with a given constant current, one can observe how the attenuation and frequency of the oscillating system vary; this is equivalent to varying the restoring force of the spring. This interaction can be represented as a quartic potential energy well, where the magnet executes its oscillations [7, 11]. By turning off the first coil’s constant current and exciting the second coil with an alternating current, one can monitor the forced oscillations of the system separately.

We should mention that as the oscillation amplitudes get larger than 2 cm, the spring begins to execute torsion oscillations (like a Wilberforce pendulum [2]). We managed to reduce these undesirable oscillations in the following simple way: an auxiliary 2 cm wide strip of a transparency sheet (cellulose acetate) was arranged to hang (like a smooth and flexible arched ‘tongue’ or a hanging light curved curtain) with its lower end scotch-taped to our acetate
light-filter, while the other upper end was fixed to an auxiliary fixed support (the concavity of this arched 'tongue' always pointing upwards). This auxiliary curved acetate strip can be seen just below the bottom-right corner of the inset in figure 2. We found that the slight pressure force exerted by this auxiliary elastic strip, effectively counteracted the torsion oscillations of the spring, checking the spring torsion oscillations as the magnet oscillated up-and-down without adding any significant friction to the oscillator (long lasting free oscillations can still be observed).

3. Theoretical model

3.1. The magnet interaction with the short coil

If $i_0$ denotes the exciting constant electrical current through the small coil (of $N$ turns, length $L$ and radius $a$ shown in figure 1), then the current element $di$ along a small cylindrical coil element of height $dz$ is simply given by $di = (i_0 N/L)dz$. Using the widely known expression $dF = i_0 dli \times B$ for the magnetic force on a conducting element $dli$, carrying a current $i_0$, when placed in an induction field $B$ [13, 14], we find that the vertical force of the magnet on the cylindrical coil element is

$$dF = (2\pi a)B_p(a, z) \, di,$$

where $B_p(a, z)$ is the radial component of the magnet field at vertical coordinate $z$ and radial distance $a$, i.e. at the coil element in question.

In our recent paper [11] where the magnet interacted with a single small coil, we found that a single magnetic-dipole approximation provided a simple and convenient expression for the radial component $B_p(a, z)$ required in equation (1). In previous papers [15, 16], with more elaborate experimental setups, we found after a few experiments that that approximation was lacking, and a better model for the magnet–coil interaction was necessary. We then resorted to a two magnetic-dipole approximation which is a better approximation in two senses; firstly, the final mathematical model of the moving magnet reproduced our experimental results with greater accuracy, and second because it better represents the sensitivity of the oscillating system to small variations in the initial conditions. Therefore, we represent here the magnet as the superposition of two vertically aligned magnetic dipoles separated by the distance $2c$. For this vertical separation of the two assumed parallel dipoles, we can write the radial component of the magnet field [15, 16] as

$$B_p(a, z) = \frac{\mu_0}{4\pi} \frac{3ma}{2} \left[ \frac{z - c}{[a^2 + (z - c)^2]^{5/2}} + \frac{z + c}{[a^2 + (z + c)^2]^{5/2}} \right],$$

where $m$ is the single magnet dipole. This approximation introduces the additional parameter $2c$, which fortunately can be easily determined in a simple preliminary experiment in which the e.m.f. induced in a thin, cardboard, hand-wound auxiliary pick-up coil of a few turns is monitored with an oscilloscope (for details see the appendix of [16]).

From equations (1) and (2) we obtain the small force on the coil element of length $dz$

$$dF = \frac{(2\pi a)N i_0 \mu_0}{L} \frac{3ma}{2} \left[ \frac{z - c}{[a^2 + (z - c)^2]^{5/2}} + \frac{z + c}{[a^2 + (z + c)^2]^{5/2}} \right] dz,$$

and integrating this force along the whole coil i.e. from $z' = z - L/2$ to $z' = z + L/2$ we obtain the total magnetic force $F$ between magnet and coil.
\[ F(z) = (2\pi a) \frac{Ni_0 \mu_0 ma}{L} \frac{1}{2} \left[ \frac{1}{a^2 + \left( z - \frac{L}{2} - c \right)^2} - \frac{1}{a^2 + \left( z + \frac{L}{2} - c \right)^2} \right] \]
\[ + \frac{1}{a^2 + \left( z - \frac{L}{2} + c \right)^2} - \frac{1}{a^2 + \left( z + \frac{L}{2} + c \right)^2} \]. \quad (4)

If, for now, we restrict the analysis to small displacements of the magnet with respect to the coil horizontal mid-plane, i.e. when the magnet coordinate satisfies \( z(t) \ll L/2 \), we can find the force \( F \) on the magnet given by equation (4) in terms of the variable \( z \), by expanding the four binomials on the denominators of the rhs using a widely known procedure (see the appendix). Having done that, we obtain the following simple, nonlinear expression that represents the interaction force that the small coil applies to the magnet when excited with a dc current,

\[ F(z) = F_1 \frac{z}{a} \left( 1 - \frac{z^2}{a^2} \right), \]

where the constant force \( F_1 \) and the dimensionless coil constant \( C \) are given in the appendix. Note there, that \( F_1 \) is proportional to the dc current \( i_0 \) in the small coil.

### 3.2. Interaction between the magnet and the large coil

As mentioned above, the large coil in figure 1 is placed at a relatively large distance from the region where the magnet oscillates. It is located there deliberately (about 16 cm below the small coil) so that the force it produces on the magnet goes through a maximum, and so one can reasonably assume a constant interaction force between this large coil and the magnet in the region of the maximum (the slope of a curve is zero in the region of a maximum). The magnitude of this force can be measured easily in a prior and straightforward experiment (with the short coil dc current cut off), and in our particular setup it turned out to be of the order of a few tens of mN per each ampere in the amplitude of the oscillating excitation current in the large coil.

### 3.3. The motion equation of the oscillator

Having dealt with the two magnet–coil interactions separately, and taking into account the elastic force \( kz \) of the spring and the weak viscous force \( k_v \dot{z} \) on the moving magnet, we can now write its motion equation for small displacements \( z \) (note equation (5))

\[ M \ddot{z} + k_v \dot{z} + kz = F_1 \frac{z}{a} - F_1 C \frac{z^3}{a^3} + F_0 \cos(\gamma t). \]

\( (6) \)

Note that the first two terms in the rhs of equation (6) represent the forces applied to the magnet by the small coil when excited with a constant current, while the last term represents the variable force applied by the large coil when excited with an oscillating current of angular frequency \( \gamma \), i.e. the periodic driving force on the oscillator. The elastic constant of the spring is, of course, \( k \equiv M\omega_0^2 \). Equation (6) can be rewritten as

\[ \ddot{z} + 2\lambda \dot{z} + \omega_0^2 z = - \frac{F_1}{M a^3} z^3 + \frac{F_0}{M} \cos(\gamma t), \]

\( (7) \)

where we have introduced a parameter \( \lambda = k_v/2M \) that accounts for the friction on the cylindrical magnet. Also note that the first order term in \( z \) in the rhs has been absorbed into
the elastic constant $k$ in the lhs of the motion equation, which means a new ‘natural’ angular frequency $\omega_1$ of oscillation given by

$$\omega_1 = \omega_0 \sqrt{1 - \frac{F_1}{M \omega_0^2 a}} = \omega_0 \sqrt{1 - \alpha},$$

(8)

where we have introduced the convenient parameter $\alpha = \frac{F_1}{M \omega_0^2 a}$, proportional to the current $i_0$ (see equation (A.4)). Note that this angular frequency $\omega_1$ does not depend upon the amplitude $z_0$ of the oscillating magnet.

Equation (7) is the equation of a forced damped Duffing oscillator [2, 4, 6, 7], with its characteristic cubic term. It is important and interesting to interpret the role that the short coil plays in the magnet oscillations when excited with a constant current. This can be readily seen by discussing the two force components at the rhs of equation (6). We obtain $i_0 > 0$ when the force $F_1 (z/a)$ opposes the elastic force $-kz$ of the spring.

From our knowledge of the magnetic force that the small coil applies to the magnet, we can derive the expression for its potential energy function $U$ in the absence of the driving force. It is given by

$$U(z) = \frac{1}{2} M \omega_1^2 z^2 + \frac{F_1 C}{4a^3} z^4,$$

(9)

which is a quartic potential plotted in figure 3.

Note that when $\alpha > 1$, the quantity $\omega_1^2$ is negative (see equation (8)) and the potential function $U$ presents two minima and a maximum in between, i.e. it has two sub-wells with an unstable equilibrium point in between, the maximum of a potential barrier. In this case, the magnet will jump chaotically between the sub-wells when excited with an additional weak oscillating magnetic force (interaction with the large coil of figure 1 oscillating momentarily in either sub-well (see below section 4.3). In the case $0 < \alpha < 1$, the potential has a flattened bottom with a single stable minimum. In this second case, the magnet will show its nonlinear resonances within the single flattened well [11].
3.4. Nonlinear resonances of the oscillator

Let us begin by studying the case where $0 < \alpha < 1$. Here, we seek solutions of the motion equation (7) with the form

$$z = z_0 \cos(\omega t)$$

(as Landau and Lifshitz did, [1]). As usual the cubic term in that equation is expanded as follows:

$$z^3 = z_0^3 \cos^3(\omega t) = z_0^3 \left[ \frac{3}{4} \cos(\omega t) - \frac{1}{4} \cos(3\omega t) \right], \quad (10)$$

After neglecting the third harmonic term and replacing what remains into the motion equation (7), and considering for the time being the non-forced oscillations with negligible damping (i.e. set $F_0 = 0$, $k_v = 0$), we get what would be the resonance angular frequency $\omega$ for the large amplitude free oscillations

$$\omega = \omega_0 \sqrt{1 - \alpha + \frac{3}{4} \alpha C \left( \frac{z_0}{a} \right)^2}. \quad (11)$$

Contrary to equation (8), this angular frequency of the magnet does depend upon the oscillation amplitude $z_0$. Factoring out the term $\sqrt{1 - \alpha}$ in equation (11), and expanding the remaining square root up to first order in $z_0^2$, we get an approximate expression for the angular frequency $\omega$ of the large and un-damped free oscillations of the magnet,

$$\omega \approx \omega_1 + \frac{3\alpha C \omega_1}{8(1 - \alpha)a} z_0^2, \quad (12)$$

which, after introducing a new parameter $\kappa = \frac{3\alpha C \omega_1}{8(1 - \alpha)a}$, can be rewritten as

$$\omega \approx \omega_1 + \kappa z_0^2. \quad (13)$$

Note that in this section so far, we are dealing with three different angular frequencies, namely $\omega_0$ which is parameter related to the free oscillations of the magnet–spring system (basically it represents the elastic constant of the system), $\omega_1$ which resulted from a convenient re-arranging of the force term in the motion equation (it is the new oscillation frequency for the small free oscillation) and finally $\omega$ which includes the nonlinear effect (since it depends upon the amplitude $z_0$).

Going back to the motion equation (6) and considering only its linear terms, we obtain the widely known amplitude-versus-frequency standard resonance relation of linear damped harmonic oscillators [1, 14]

$$z_0 = \frac{F_0}{M \sqrt{\left( \omega_1^2 - \gamma^2 \right)^2 + 4\lambda^2 \gamma^2}}, \quad (14)$$

When close to resonance, we can rewrite the angular frequency $\gamma$ of the forced amplitude oscillations in terms of the oscillating system $\omega_1$, as Landau and Lifshitz did in [1]: $\gamma = \omega_1 + \varepsilon$, as well as $\gamma^2 - \omega_1^2 = (\gamma + \omega_1)(\gamma - \omega_1) = 2\omega_1 \varepsilon$. Replacing this definition of the small number $\varepsilon$, and the result $2\omega_1 \varepsilon$ into equation (14), we can obtain the amplitude of the stationary regime oscillations

$$z_0 = \frac{F_0}{2M \omega_1 \sqrt{\varepsilon^2 + \lambda^2}}. \quad (15)$$

Considering the large amplitude forced oscillations we may redefine the departure $\varepsilon$ from resonance (see equation (13)) as

$$\gamma = \omega_1 + \varepsilon_0 + \varepsilon. \quad (16)$$
Finally by solving for $\varepsilon$ in equation (15) and replacing the result into equation (16), we obtain the sought relation between the oscillation frequency and the amplitude of the large amplitude oscillations

$$\gamma = \omega_1 + z_0^2 \pm \sqrt{\left(\frac{F_0}{2M\omega_1 z_0}\right)^2 - \lambda^2}. \quad (17)$$

This equation is the equivalent of the standard resonance equation (14) for the large driven amplitude oscillations of our system.

Plotting the oscillation amplitude $z_0$ versus the oscillator temporal frequency $\gamma/(2\pi)$ as given by our equation (17), we get figure 4. This figure represents a set of resonance curves of our oscillator for a few values of the force amplitude $F_0$ (equivalent to different amplitudes of the ac current in the large coil of the present oscillating system). It can be seen how the resonance peaks gradually shift to one side as the amplitude of the oscillations increases, the resonance curves eventually folding over. For our oscillator, it can be shown that this tilting or fold-over effect begins for a large coil current of amplitude 0.10 A (see [1] ibid). The dotted parabolic arch in the figure corresponds to the sum $\omega_1 + \frac{\omega_0\omega_1}{\sqrt{1-\omega_0^2}} z_0^2$ in equation (12); it is called the spine of the resonances, the locus of the peaks of the resonances curves.

4. Experiments and results

To validate the analytical model developed in section 3, we have performed a set of four basic experiments that illustrate the nonlinear features of our oscillator. Before this we performed a set of straightforward experiments either to calibrate or to measure relevant parameters of the system, or simply to set it properly.

4.1. The excitation dc current and the parameter alpha

In this first experiment we measured the angular frequency $\omega_1$ of the free damped oscillations of the magnet for different values of the small coil exciting current $i_0$, i.e. for different values of the dimensionless coil excitation parameter $\alpha = \frac{F_1}{M\omega_1 z_0}$ and for no current at all in the large coil. To this effect we manually set the magnet oscillating with relatively small amplitudes ($z_0$ less than $L/3$, which means 4–7 mm in our oscillator) while keeping the excitation parameter $\alpha$ less than one, i.e. $0 < \alpha = \frac{F_1}{M\omega_1 z_0} < 1$. The angular frequency was then read directly from
Figure 5. Angular frequency of the oscillator plotted against the small coil excitation parameter $\alpha$ (that is proportional to the dc current $i_0$ in the range $-0.5 \text{ A} < i_0 < 0.5 \text{ A}$).

the stored oscillations in the digital oscilloscope (see section 2). Figure 5 shows plots for both our experimental results and the function predicted by our theoretical model $\omega_1 = \omega_0 \sqrt{1 - \alpha}$ (see equation (8)). Note that we plot normalized angular frequency values $\omega_1/\omega_0$ versus the excitation parameter $\alpha$ (corresponding to $i_0$ in the range $-0.5$ to $+0.5$ A). Note that $\omega_0$ is the eigenfrequency of the magnet–spring system when the small coil current is zero. There is good agreement between the experimental data and our model prediction.

The normal regime of oscillation is when $0 < \alpha < 1$. When $\alpha > 1$, the system oscillates in a broad well with two sub-wells at the bottom (see figure 3), leading to bifurcation and chaotic motion as will be shown below. When $\alpha < 0$, the magnetic forces of the coil on the magnet have the same direction of the elastic force of the spring, i.e. this amounts to a stiffer spring and therefore the angular frequency increases and the nonlinear term of the motion equation becomes less significant.

4.2. Tilted resonances

After launching the magnet into small, initial amplitude oscillations, we scanned the temporal frequency of the large coil ac excitation current in the range 0.2–1.5 Hz, while keeping constant the electrical dc current $i_0$ through the small coil. We then observed and stored the varying amplitude of the nonlinear oscillations in the digital oscilloscope. Our experimental results are plotted in figures 6(a) and (b) jointly, with the theoretical curves predicted by equation (17).

The resonance curve in figure 6(a) was plotted for the small coil dc current $i_0 = 0.40 \text{ A}$ and a large-coil ac excitation current amplitude $i_C = 0.10 \text{ A}$, while the resonance curve in figure 6(b) was plotted for $i_0 = 0.40 \text{ A}$, and $i_C = 0.17 \text{ A}$ (which means a force amplitude $F_0 = 2 \text{ mN}$, on the oscillating magnet and a friction coefficient $\lambda = 0.3 \text{ s}^{-1}$ in the latter case).

In figure 6, the tilting or folding-over of the resonance curves is more pronounced, as expected, for a larger current in the large coil, i.e. for a larger driving force on the oscillating magnet. When obtaining these two resonances curves, the frequency of the ac current exciting the large coil was first gradually increased (crosses in the plot) and then gradually decreased (small circles) in the range 0.4–1.0 Hz. Note then the discontinuous jumps [9] in amplitude at about 0.85 Hz (crosses) and at 0.73 Hz (circles) in figure 6(b). The agreement between the experimental data and the curves is acceptable. The departure of the data points from the theoretical curves is of the order of $\sim 2 \text{ mm}$, i.e. an error of $\sim 20\%$. The data points lie mostly below the curves and we believe it to be because of an additional friction effect of non-viscous origin not yet considered in our mathematical model.
Figure 6. Experimental resonance curves (a) for an ac current amplitude $i_C = 0.10\, \text{A}$ in the large coil driving the magnet oscillation. The crosses represent experimental data when increasing the coil current frequency from 0.4 to 1.0 Hz, the small circles when decreasing it. (b) for an ac current amplitude of 0.17 A in the large coil. Note the discontinuous jumps in oscillation amplitude at 0.73 Hz and at 0.85 Hz.

Figure 7. Time-series of the oscillator in the two sub-wells (for $i_0 = 0.7\, \text{A}$). The upper oscillations correspond to motion in one sub-well (centred at 6 mm) and the lower oscillations to the other sub-well (centred at –6 mm). The large coil current $i_C$ is set so that the magnet just overcomes the unstable maximum of the main well (onset of chaotic motion).

4.3 Time series, phase plane plot and chaos

Let us now consider the case when $\alpha > 1$. When dealing with nonlinear oscillations, it has been found useful to represent the motion of the system by simply plotting the oscillator position versus the time in a so-called time series plot, and figure 7 is such a plot for our oscillator. By increasing the dc current in the small coil, the single flattened potential well evolves to one with two well differentiated sub-wells at the bottom, separated by a maximum. Then by exciting the large coil with an oscillating current of sufficient amplitude to make the magnet
overcome the potential barrier that separates the two sub-wells, it is possible to observe the magnet shifting from one to the other sub-well. In the oscillations plotted above in figure 7, the unstable equilibrium point at \( z = 0 \) actually takes place about the point of vertical ordinate \( z = +6 \) mm, while the lower ones are centred at about \( z = -6 \) mm, i.e. these oscillations represent the motion of the magnet inside the two sub-wells of the oscillator’s quartic potential. It can also be seen in how the oscillator regularly shifts (bifurcates) from one sub-well to the other.

A second useful representation of nonlinear oscillators is the phase-plane plot obtained by plotting the object speed \((dz/dt)\) versus the position \((z)\) of the object, as shown in figure 8 for the present case. The magnet position data was taken directly from the stored data in the digital scope (figure 7), while the magnet speed was obtained by taking the derivative of the magnet position (previously averaged over short position intervals of the same size). To avoid a tangled phase-plane representation, we restricted the plotting range in figure 8 to the magnet motion within the first 20 s of the time-series plot of figure 7. It can be seen in the phase plot that there is an attractor bounded at both sides of the magnet equilibrium position at \( z = 0 \). The orbits centred at about \( z = +6 \) mm correspond to the upper portion of the time-series plot, while the orbits centred at about \( z = -6 \) mm correspond to the lower portion of the time-series plot. The lines connecting the orbits correspond to the wiggly lines connecting the upper and lower oscillations of the time-series plot in figure 7.

6. Discussion and final comments

We have presented a forced magneto-mechanical, nonlinear oscillator that consists of a ferrite magnet hanging from a spring oscillating in the magnetic fields of two hand-made coaxial electromagnetic coils; the first excited with a dc current, the second with an ac current. It is a very simple oscillating system that requires a minimum of parts and ordinary laboratory equipment (in fact a low-voltage power supply, a signal generator followed by a current amplifier, and a digital oscilloscope to measure both the oscillation amplitudes of the magnet and the oscillations’ angular frequencies, the latter with great accuracy. This driven nonlinear oscillator is much simpler and easier to setup than many of the nonlinear oscillators presented in the literature in the last two decades, and has a small number of components. It is however capable of showing interesting nonlinear dynamic effects. We also derived a theoretical model of the oscillator using classical electrodynamics and Newton’s laws. The resulting motion equation of the magnet contains a cubic term in the oscillator coordinate; our system is just a forced and damped cubic Duffing-type oscillator. This equation has been solved analytically.
and we have been able to derive the nonlinear resonance curves of the oscillator in terms of the two coils’ currents. A set of basic experiments revealing some of these effects and properties was presented. The results of the experiments have been presented and interpreted correctly. We have a case of an object that oscillates in a quartic potential which can be seen evolving from a flattened-bottom well to one with two sub-wells separated by a point of unstable equilibrium. When driven with an external variable frequency generator, our nonlinear oscillator will show the nonlinear effects typical of such oscillators. By adjusting the amplitude of the oscillating current of the large coil, we can force the magnet to just overcome the energy potential barrier the oscillator and it then becomes chaotic. The oscillations of the system for the parameter $\alpha < 0$ are to be studied later. The nonlinear oscillator presented here is very suitable for a senior laboratory experiment in nonlinear dynamics.

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Appendix

By expanding the four binomials in the denominators of equation (4) up to second order (see the appendix in [11]) and after simplifying the results, one obtains

$$F = \frac{3Nmi_0}{2a^2\left[1 + \left(\frac{L}{2}\right)^2\right]^{5/2}} \left\{ \frac{z}{a} \left(1 - \frac{3Dc^2}{a^2}\right) - D\frac{z^3}{a^4} \right\}, \quad \text{(A.1)}$$

where the constant $D$ is given by

$$D = \frac{5\left(3 - 4\left(\frac{L}{2}\right)^2\right)}{6\left(1 + \left(\frac{L}{2}\right)^2\right)^2}. \quad \text{(A.2)}$$

Factoring out the first term $\frac{z}{a} \left(1 - \frac{3Dc^2}{a^2}\right)$ from the curly bracket in equation (A.3) one gets

$$F = \frac{3Nmi_0}{2a^2\left[1 + \left(\frac{L}{2}\right)^2\right]^{5/2}} \left\{ \frac{z}{a} \left(1 - \frac{3Dc^2}{a^2}\right) \frac{z}{a} \left(1 - D\left(1 - \frac{3Dc^2}{a^2}\right)^{-1}\frac{z^2}{a^2}\right) \right\}. \quad \text{(A.3)}$$

Finally, defining the constant force $F_1$ and the coil constant $C$ by

$$F_1 = \frac{3Nmi_0}{2a^2\left[1 + \left(\frac{L}{2}\right)^2\right]^{5/2}} \left(1 - \frac{3Dc^2}{a^2}\right), \quad \text{(A.4)}$$

$$C = D\left(1 - \frac{3Dc^2}{a^2}\right)^{-1}. \quad \text{(A.5)}$$

and replacing them in equation (A.3) we get our equation (5) in the main text.

References

[3] Holmes P and Holmes C 1981 Second order averaging and bifurcations to subharmonics and bifurcations in Duffing’s equation J. Sound Vib. 78 161–4


