THE L^p-CONTINUITY OF WAVE OPERATORS FOR FRACTIONAL ORDER SCHRÖDINGER OPERATORS

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ABSTRACT. We consider fractional Schrödinger operators $H = (-\Delta)^\alpha + V(x)$ in n dimensions with real-valued potential V when $n > 2\alpha$, $\alpha > 1$. We show that the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$ under conditions on the potential that depend on n and α analogously to the case when $\alpha \in \mathbb{N}$. As a consequence, we deduce a family of dispersive and Strichartz estimates for the perturbed fractional Schrödinger operator.

1. INTRODUCTION

We study the non-local fractional Schrödinger equation

$$i\psi_t = (-\Delta)^\alpha \psi + V\psi, \qquad x \in \mathbb{R}^n, \qquad \alpha > 1, \quad \alpha \notin \mathbb{N}.$$

We consider spatial dimensions $n > 2\alpha$, and V is a real-valued, decaying potential. When $\alpha \notin \mathbb{N}$, the non-local operator $(-\Delta)^\alpha$ is defined via the Fourier multiplier $|\xi|^{2\alpha}$, that is $(-\Delta)^\alpha f = \mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{f}(\xi))$. We denote the free fractional Schrödinger operator by $H_0 = (-\Delta)^\alpha$, and the perturbed operator by $H = (-\Delta)^\alpha + V(x)$.

Similar to the integer order Schrödinger operators, for the potentials we consider there is a Weyl criterion and $\sigma_{ac}(H) = \sigma_{ac}(H_0) = [0, \infty)$. When $\alpha \neq 1$, decay of the potential is generally not sufficient to ensure the lack of eigenvalues embedded in the continuous spectrum for the fractional order operators. In [10], Cuenin constructs examples where embedded eigenvalues can occur for generalized Schrödinger operators. On the other hand, Ishida, Lőrinczi and Sasaki provided conditions on the potential when $0 < \alpha < 2$ in [37] for which H has no embedded eigenvalues. We leave the lack of embedded eigenvalues as an overarching assumption.

We study the L^p boundedness of the wave operators, which are defined by

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}.$$

As in the classical Schrödinger operators, see the work of Agmon, [1], Hörmander, [36] and Schechter, [52], sufficient decay of the potential at infinity to ensures that the wave operators exist and are

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asymptotically complete, [38, 62]. In particular we have the intertwining identity

$$f(H)P_{ac}(H) = W_{\pm} f((-\Delta)^{\alpha}) W_{\pm}^{*}. \tag{1}$$

Here $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace of H, and f is any Borel function. Asymptotic completeness and lack of embedded eigenvalues for magnetic fractional equations were studied in [54]. For a more detailed discussion of the existence and asymptotic completeness see [62].

Fractional Schrödinger equations have garnered interest in the physics literature, see for example [44, 45], where the Feynman path integral is taken over Lévy flight paths in place of Brownian paths. For applications in optics see [46], further applications are considered in [33]. Mathematically, one can view fractional Schrödinger equations as models of nonlocal dispersive equations.

We use the notation $\langle x\rangle$ to denote $(1+|x|^2)^{\frac12}$, $\mathcal{F}(f)$ or $\widehat{f}$ to denote the Fourier transform of f. We write $A \lesssim B$ to say that there exists a constant C with $A \leq CB$, and write $a- := a-\epsilon$ and $a+ := a+\epsilon$ for some $\epsilon > 0$ throughout the paper. We use the norm $\|f\|_{H^\delta} = \|\langle\cdot\rangle^{\delta}\widehat{f}(\cdot)\|_2$. We first state a small potential result.

Theorem 1.1. *Fix $\alpha > 1$ and let $n > 2\alpha$. Assume that the V is a real-valued potential on $\mathbb{R}^n$ and fix $0 < \delta \ll 1$. Then $\exists C = C(\delta, n, \alpha) > 0$ so that the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$, provided that*

i) $\big\|\langle\cdot\rangle^{\frac{4\alpha+1-n}{2}+\delta} V(\cdot)\big\|_2 < C$ *when* $2\alpha < n < 4\alpha - 1$,
ii) $\big\|\langle\cdot\rangle^{1+\delta} V(\cdot)\big\|_{H^\delta} < C$ *when* $n = 4\alpha - 1$,
iii) $\big\|\mathcal{F}(\langle\cdot\rangle^{\sigma} V(\cdot))\big\|_{L^{\frac{n-1-\delta}{n-2\alpha-\delta}}} < C$ *for some* $\sigma > \frac{2n-4\alpha}{n-1-\delta} + \delta$ *when* $n > 4\alpha - 1$.

The assumptions on the potential are the generalizations of the $\alpha = m \in \mathbb{N}$ case studied in [16] obtained by bounding the contribution of the Born series terms in Section 2 below. There are technical hurdles to overcome to adapt the argument to the non-local fractional Schrödinger operators, and the analysis is rather delicate.

Furthermore, one may remove the smallness assumption above provided V decays sufficiently at spatial infinity. We write $n_\star$ to denote $n+4$ if n is odd and $n+3$ if n is even. We define zero energy to be regular if there are no non-trivial distributional solutions to $H\psi = 0$ with $\langle x\rangle^{-\alpha-}\psi(x) \in L^2(\mathbb{R}^n)$ when $2\alpha < n \leq 4\alpha$ and $\psi \in L^2(\mathbb{R}^n)$ when $n > 4\alpha$, which correspond to resonances or eigenvalues respectively, see [15]. We show

Theorem 1.2. *Fix $\alpha > 1$ and let $n > 2\alpha$. Assume that the V is a real-valued potential on $\mathbb{R}^n$ so that*

i) $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > n_\star$,
ii) $\|\langle\cdot\rangle^{1+} V(\cdot)\|_{H^{0+}} < \infty$ *when* $n = 4\alpha - 1$,

iii) for some $0<\delta\ll 1$ *and* $\sigma>\frac{2n-4\alpha}{n-1-\delta}$, $\|\mathcal{F}(\langle\cdot\rangle^\sigma V(\cdot))\|_{L^{\frac{n-1-\delta}{n-2\alpha-\delta}}}<\infty$ *when* $n>4\alpha-1$,

iv) $H=(-\Delta)^\alpha+V(x)$ *has no positive eigenvalues and zero energy is regular.*

Then, the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ *for all* $1\le p\le\infty$.

These results are, to the best of our knowledge, the first results studying L^p-boundedness of the wave operators for the non-local fractional Schrödinger operators.

From the intertwining identity (1) one may obtain L^p-based mapping properties for the more complicated, perturbed operator $f(H)P_{ac}(H)$ from the simpler free operator $f((-\Delta)^\alpha)$. The boundedness of the wave operators on $L^p(\mathbb{R}^n)$ for any choice of $p\ge 2$ with the function $f(\cdot)=e^{-it(\cdot)}$ yield the family of dispersive estimates

Corollary 1.3. *Under the conditions of Theorem 1.1 or 1.2, for any* $1\le p\le 2$ *we have the following family of dispersive bounds*

$$\|e^{-itH}P_{ac}(H)\|_{L^p\to L^{p'}}\lesssim |t|^{\frac{n}{\alpha}(\frac12-\frac1p)}, \tag{2}$$

where p' *is the Hölder conjugate of* p.

In particular in all dimensions $n>2\alpha$, we have the global bounds

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty}\lesssim |t|^{-\frac{n}{2\alpha}},$$

which extends the recent work of the authors, [15], to dimensions $n>4\alpha-1$. Another consequence, following the seminal work of Ginibre and Velo, [28], is a family of Strichartz estimates:

Corollary 1.4. *Under the conditions of Theorem 1.1 or 1.2, we have*

$$\|e^{-itH}P_{ac}(H)f\|_{L^q_tL^r_x}\lesssim\|f\|_{L^2},\quad \frac{2}{q}=\frac{n}{\alpha}\left(\frac12-\frac1r\right),\quad 2\le r<\infty.$$

Noting that the Fourier transform of $e^{i|\xi|^{2\alpha}}|\xi|^{\gamma-n}$ is bounded when $\frac12<\alpha$ and $0<\gamma\le n\alpha$, by scaling the free operator satisfies the bounds

$$\|e^{-it(-\Delta)^\alpha}(-\Delta)^{\frac{\gamma-n}{2}}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{\gamma}{2\alpha}}.$$

We have the following corollary. Similar bounds were proved in [14] for integer $\alpha\in(\frac{n}{4},\frac{n}{2})$.

Corollary 1.5. *Under the conditions of Theorem 1.1 or 1.2, for any* $0<\gamma\le n\alpha$ *we have the following family of dispersive bounds*

$$\|e^{-itH}H^{\frac{\gamma-n}{2\alpha}}P_{ac}(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{\gamma}{2\alpha}}.$$

This work is inspired by previous work of the first and third authors, [16, 17], studying the boundedness of the wave operators when $\alpha = m \in \mathbb{N}$. There are several technical challenges in this adaptation, such as the lack of a "splitting identity" that allows one to explicitly equate the integer order Schrödinger resolvents to the more-well known second order resolvent, see (4) below.

There has been substantial work on the $L^p(\mathbb{R}^n)$ boundedness of the wave operators when $\alpha = m \in \mathbb{N}$, with recent growth in the literature when $m > 1$. The first higher order result, when $(m,n) = (2,3)$ was established by the second and third authors in 2020, [27]. The first and second authors extended the range to (m,n) for all $n > 2m$ in [16, 17]. Mizutani, Wan, and Yao studied the case of $(m,n) = (2,1)$ in [48], and studied the endpoints and effect of threshold resonances in the $(m,n) = (2,3)$ case in [49, 50]. Galtbayar and Yajima consider the case of $(m,n) = (2,4)$ in [24].

The study of the wave operators when $m > 1$ partially built upon work on dispersive estimates. Feng, Soffer, Wu and Yao proved "local dispersive estimates" on the solution operator as a map between weighted L^2-based in [22]. The third author and Toprak, [31], along with the first author, [20], provided "global dispersive estimates" considering the solution operator as a map from L^1 to L^∞ for the fourth order operator in dimensions $n = 4$ and $n = 3$ respectively. The authors recently proved dispersive estimates for scaling-critical potentials when $2m < n < 4m$, [14].

The wave operators for the usual Schrödinger operator $-\Delta + V$, when $m = 1$ are well-studied, beginning with the pioneering works of Yajima, [55, 56, 57]. Which inspired further work when $m = 1$ in all dimensions $n \geq 1$, see [40, 41, 11, 47, 58, 59] for example. On $\mathbb{R}^3$, Beceanu and Schlag obtained detailed structure formulas for the wave operators, [2, 3, 4]. The L^2 existence and other properties of the higher order wave operators have been studied by many authors, including Agmon [1], Kuroda [42, 43], Hörmander [36], and Schechter, [52, 53].

There has been much interest in non-linear fractional Schrödinger equations, see for example [7, 35, 32, 21, 51, 12, 5], studying well-posedness, blow-up and scattering. However, the linear analysis is more limited with results focusing on the free equation $iu_t = (-\Delta)^s u$. Cho, Ozawa, and Xia studied dispersive and Strichartz estimates for the free operator assuming initial data in distorted Besov spaces, [9]. Further study of Strichartz estimates for related operators may be found in [8, 30], for example. To the best of our knowledge, the only result on dispersive estimates for a perturbed equation is that of the authors in [15].

Our analysis relies on careful study of the resolvent operators, which are defined by $\mathcal{R}_V(\lambda) = ((-\Delta)^\alpha + V - \lambda)^{-1}$ and $\mathcal{R}_0(\lambda) = ((-\Delta)^\alpha - \lambda)^{-1}$. Our usual starting point to study the wave operators is the stationary representation

$$W_+ u = u - \frac{1}{2\pi i}\int_0^\infty \mathcal{R}_V^+(\lambda) V[\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)]u \, d\lambda,$$

here the superscripts '+' and '-' denote the usual limiting values as λ approaches the positive real line from above and below, [15]. Since the identity operator is bounded on L^p, we need only bound the second term involving the integral. It is convenient to make the change of variables $\lambda \mapsto \lambda^{2\alpha}$ and consider the integral kernel of the operator

$$-\frac{\alpha}{\pi i}\int_0^\infty \lambda^{2\alpha-1}\mathcal{R}_V^+(\lambda^{2\alpha})V[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2\alpha})\,d\lambda. \tag{3}$$

Our result in Theorem 1.1 follows by using resolvent identities to expand $\mathcal{R}_V^+$ in an infinite series and directly summing the series. To remove the smallness assumption to show that the operator defined in (3) extends to a bounded operator on L^p requires different strategies in the low ($0 < \lambda \ll 1$) and high ($\lambda \gtrsim 1$) energy regimes. To delineate these cases, we use the even, smooth cut-off function χ with $\chi(\lambda) = 1$ for $|\lambda| < \lambda_0$ for some sufficiently small $\lambda_0 \ll 1$, and $\chi(\lambda) = 0$ for $|\lambda| > 2\lambda_0$, as well as the complimentary cut-off $\widetilde{\chi}(\lambda) = 1 - \chi(\lambda)$.

When $\alpha = m \in \mathbb{N}$, we have the splitting identity for $z \in \mathbb{C} \setminus [0, \infty)$, (c.f. [22])

$$\mathcal{R}_0(z)(x,y) := ((-\Delta)^m - z)^{-1}(x,y) = \frac{1}{mz^{1-\frac{1}{m}}}\sum_{\ell=0}^{m-1}\omega_\ell R_0(\omega_\ell z^{\frac{1}{m}})(x,y) \tag{4}$$

where $\omega_\ell = \exp(i2\pi\ell/m)$ are the m^{th} roots of unity, $R_0(z) = (-\Delta - z)^{-1}$ is the usual (2^{nd} order) Schrödinger resolvent. For the fractional operators, when $\alpha \notin \mathbb{N}$, we lack this explicit relationship to the $m = 1$ Schrödinger resolvents. We instead utilize the representations developed in [15] stated in Proposition 3.3 below as well as directly bounding Fourier multipliers corresponding to the Born series in Theorem 2.1 below.

We note that the different assumptions on the potential we impose based on the size of n versus α in Theorems 1.1 and 1.2 are natural. Smoothness of the potential is required for the integer order Schrödinger operator in high dimensions since the kernel free resolvent $R_0^\pm(\lambda^2)$ grows like $\lambda^{\frac{n+1}{2}-2m}$ as the spectral parameter $\lambda \to \infty$. This causes the $L^1 \to L^\infty$ dispersive estimates to fail in dimensions greater than $4m-1$ without some measure of smoothness on the potential, see the counterexample constructed by the second author and Visan [29], later extended by the authors to the higher order case, [13]. In particular, when $n > 4m-1$ one can construct a compactly supported potential $V \in C^\alpha(\mathbb{R}^n)$ for all $0 \leq \alpha < \frac{n+1}{2} - 2m - \frac{n}{p}$ for which the wave operators are unbounded on $L^p(\mathbb{R}^n)$ for $\frac{2n}{n-4m+1} < p \leq \infty$. As in the integer order analysis, [55, 16], we impose a condition on the $\mathcal{F}L^r$ norm of the potential, which requires sufficient smoothness. The ϵ-smoothness requirement in the case $n = 4\alpha - 1$ could be an artifact of our methods.

We assume that zero energy is regular, that is there are no threshold resonances or eigenvalues. These can be characterized in terms of distributional solutions to $H\psi = 0$, with ψ in weighted $L^2(\mathbb{R}^n)$ spaces, see section 8 of [22] for the integer order case and [15] for the fractional case. The effect of zero energy resonances or eigenvalues on the L^p-boundedness of the integer order wave operators is well-studied. In the classical $m = 1$ case the wave operators are generically bounded on $1 < p < \frac{n}{2}$ in the presence of a threshold obstruction when $n \geq 3$, while further orthogonality conditions allows one to obtain a larger range, [59, 26, 60, 61]. In the higher order case, $m \in \mathbb{N}$ and $m > 1$, the wave operators are bounded for $1 < p < \frac{n}{2m}$ in the presence of zero energy eigenvalues when $n > 4m$, [18]. In lower dimensions, there is a more complicated resonance structure, see [6] for odd n. In the case of an eigenvalue only, if the zero energy eigenspace is orthogonal to $x^\alpha V(x)$ for multi-indices $|\alpha| < k_0$, the wave operators are bounded on $1 \leq p < \frac{n}{2m-k_0}$ and one can recover $p = \infty$ if $k_0 > 2m$, [19]. One expects analogous results would hold for the fractional operators in the presences of zero energy obstructions, we plan to address this in a future paper.

The paper is organized as follows. In Section 2 we begin by analyzing the Born series terms that arise by iterating the resolvent identity for the perturbed resolvent in the stationary representation, (3). Next in Section 3, we control the contribution of the tail of the Born series in the low energy regime, when the spectral parameter λ is in a neighborhood of zero. In Section 4, we provide the technical arguments about inverse operators in the low energy regime to complete the low energy analysis. In Section 5 we control the remainder in the high energy regime, when $\lambda \gtrsim 1$.

2. Born Series

By iterating the resolvent identity, one has the expansion

$$\mathcal{R}_V(z) = \sum_{J=0}^{2\ell} \left[\mathcal{R}_0(z)(-V\mathcal{R}_0(z))^J\right] - (\mathcal{R}_0(z)V)^\ell \mathcal{R}_V(z)(V\mathcal{R}_0(z))^\ell. \tag{5}$$

Consider the contribution of an arbitrary summand in the Born series to (3),

$$W_J := (-1)^{J+1}\frac{1}{2\pi i}\int_0^\infty (\mathcal{R}_0^+(\lambda)V)^J[\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)]\, d\lambda.$$

In this section, we modify the proof in [16], which was inspired by Yajima's work at [55] for the classical Schrödinger, to control the Born series terms for the fractional Schrödinger operators. We prove that W_J extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$:

Theorem 2.1. *Fix $\alpha > 1$, a natural number $n > 2\alpha$, $1 \leq p \leq \infty$, and $0 < \delta \ll 1$. Then $\exists C = C(\delta, n, \alpha) > 0$ so that for $2\alpha < n < 4\alpha - 1$, we have*

$$\|W_J\|_{L^p \to L^p} \leq C^J \|\langle \cdot \rangle^{\frac{4\alpha+1-n}{2}+\delta} V(\cdot)\|_{L^2}^J,$$

for $n = 4\alpha - 1$, we have

$$\|W_J\|_{L^p \to L^p} \leq C^J \|\langle x \rangle^{1+\delta} V\|_{H^\delta}^J,$$

for $n > 4\alpha - 1$, we have

$$\|W_J\|_{L^p \to L^p} \leq C^J \|\mathcal{F}(\langle x \rangle^{\frac{2n-4\alpha}{n-1-\delta}+\delta} V)\|^J_{L^{\frac{n-1-\delta}{n-2\alpha-\delta}}}.$$

In what follows we will ignore most implicit constants; the factor C^J accounts for their contribution depending on n, α. The small potential result, Theorem 1.1, follows from these inequalities.

As in [55, 16], we bound the adjoint operator $Z_J = W_J^*$ on L^p, which for fixed $f \in \mathcal{S}$ is defined by

$$Z_J f(x) = \lim_{\epsilon_1 \to 0^+} \cdots \lim_{\epsilon_J \to 0^+} \lim_{\epsilon_0 \to 0^+} Z_{J,\vec{\epsilon},\epsilon_0} f(x), \tag{6}$$

where

$$Z_{J,\vec{\epsilon},\epsilon_0} f(x) := \frac{1}{2\pi i} \int_{\mathbb{R}} \big[\mathcal{R}_0(\lambda - i\epsilon_0) V \mathcal{R}_0(\lambda + i\epsilon_1) \cdots V \mathcal{R}_0(\lambda + i\epsilon_J) f\big](x) d\lambda.$$

As in [55], it suffices to prove that the limit above exists in L^p and the bounds stated in the theorem hold for $f \in \mathcal{S}$ and $\widehat{V} \in C_0^\infty$. Following the steps in page 7 and 8 of [16], we write

$$\begin{aligned} Z_J f(x) & \\ = \lim_{\epsilon_1 \to 0^+} \cdots \lim_{\epsilon_J \to 0^+} \int_{\mathbb{R}^n} T^\alpha_{k_1,\epsilon_1} \bigg\{ \int_{\mathbb{R}^n} T^\alpha_{k_2,\epsilon_2} \bigg\{ \cdots \int_{\mathbb{R}^n} K_J(k_1, k_2, \dots, k_J) T^\alpha_{k_J,\epsilon_J} f_{k_J}\, dk_J \bigg\} \cdots \bigg\} dk_2 \bigg\} dk_1, \end{aligned} \tag{7}$$

where $K_J(k_1, k_2, \dots, k_J) := \prod_{j=1}^J \widehat{V}(k_j - k_{j-1})$ (with $k_0 := 0$) and $f_{k_J}(x) := e^{ik_J \cdot x} f(x)$, and

$$T^\alpha_{k,\epsilon} f = \mathcal{F}^{-1}\bigg(\frac{\widehat{f}(\xi)}{|\xi - k|^{2\alpha} - |\xi|^{2\alpha} - i\epsilon} \bigg). \tag{8}$$

Accordingly, we need to understand the operators $T^\alpha_{k,\epsilon}$. Let

$$p_\omega(\xi) := \frac{|\xi - \omega|^2 - |\xi|^2}{|\xi - \omega|^{2\alpha} - |\xi|^{2\alpha}}, \text{ where } \omega = \frac{k}{|k|} \in S^{n-1}. \tag{9}$$

Unlike the case when $\alpha \in \mathbb{N}$, we cannot neatly factor here. We therefore have

$$T^\alpha_{k,\epsilon} f = \frac{1}{2i|k|^{2\alpha-1}} \mathcal{F}^{-1}\bigg(\frac{p_\omega(\xi/|k|)\widehat{f}(\xi)}{-\frac{i|k|}{2} + i\omega \cdot \xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2\alpha-1}}} \bigg).$$

It is easy to see that $p_\omega(\xi) \geq 0$, in fact, the proof of Lemma 2.2 below implies that $p_\omega(\xi) \approx \langle \xi \rangle^{2-2\alpha} > 0$. Writing

$$\frac{1}{-\frac{i|k|}{2} + i\omega \cdot \xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2\alpha-1}}} = -\int_0^\infty e^{-\frac{i|k|t}{2} + it\omega\cdot\xi} e^{-\frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2\alpha-1}} t} dt,$$

we obtain

$$\mathcal{F}^{-1}\bigg(\frac{p_\omega(\xi/|k|)\widehat{f}(\xi)}{-\frac{i|k|}{2} + i\omega \cdot \xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2\alpha-1}}} \bigg)(x) = -\int_0^\infty e^{-\frac{i|k|t}{2}} h_{k,\frac{\epsilon t}{2|k|^{2\alpha-1}}} * f(x + t\omega) dt,$$

where $*$ denotes convolution and

$$h_{k,\epsilon} = \mathcal{F}^{-1}\Big(p_\omega(\xi/|k|) e^{-\epsilon p_\omega(\xi/|k|)} \Big).$$

With this notation, we have

$$T^{\alpha}_{k,\epsilon} f(x) = \frac{i}{2|k|^{2\alpha-1}} \int_0^\infty \int_{\mathbb{R}^n} e^{-i|k|t/2} h_{k,\frac{\epsilon t}{2|k|^{2\alpha-1}}}(y) f(x-y+t\omega)\, dy\, dt.$$

To study the limit as $\epsilon \to 0^+$, we need the following lemma:

Lemma 2.2. *We have the following bounds (with $k = s\omega, s > 0, \omega \in S^{n-1}$)*

$$\big\| \sup_{\epsilon>0} h_{k,\epsilon} \big\|_{L^1} \lesssim 1,$$

$$\big\| \sup_{\epsilon>0} |\partial_s^j h_{s\omega,\frac{\epsilon}{s^{2\alpha-1}}}| \big\|_{L^1} \lesssim s^{-j}, \;\; j = 1, 2.$$

Furthermore, $h_{k,\epsilon}$ converges to $h_k := h_{k,0}$ and $\partial_s^j h_{s\omega,\frac{\epsilon}{s^{2\alpha-1}}}$ converges to $\partial_s^j h_k$ as $\epsilon \to 0$ a.e. and in L^1, and h_k satisfies the same bounds above.

To prove this lemma, we need the following lemmas from [15]:

Lemma 2.3. *If g compactly supported on $\mathbb{R}^n$, and is smooth on $\mathbb{R}^n \setminus \{0\}$ with $|\nabla^N g(\xi)| \lesssim |\xi|^{\gamma-N}$ for some $\gamma > -n$ and $N = 0, 1, \dots$ for $\xi \neq 0$. Then $|\nabla^j \widehat{g}(x)| \lesssim \langle x \rangle^{-n-\gamma-j}$, j=0,1,2,... In particular, $\widehat{g} \in L^1$ if $\gamma > 0$.*

Lemma 2.4. *Let g be a smooth function, supported away from zero on $\mathbb{R}^n$, that satisfies $|\nabla^N g(\xi)| \lesssim |\xi|^{\gamma-N}$ for some $\gamma < 0$ and $N = 0, 1, 2, \dots$. Then $\widehat{g}$ is a smooth function on $\mathbb{R}^n \setminus \{0\}$ satisfying*

$$|\nabla^j \widehat{g}(x)| \lesssim \begin{cases} |x|^{-\gamma-n-j} & \text{if } \gamma + j > -n, \\ \langle \log|x| \rangle & \text{if } \gamma + j = -n, \\ 1 & \text{if } \gamma + j < -n. \end{cases}$$

Morever for $|x| \gtrsim 1$, $|\nabla^j \widehat{g}(x)| \lesssim |x|^{-M}$ for all M, j.

Proof of Lemma 2.2. We first prove the claims for h_k. Note that

$$h_{s\omega}(x) = s^n \mathcal{F}^{-1} p_\omega(xs) = s^n h_\omega(sx)$$

Therefore $\|h_{s\omega}\|_{L^1} = \|h_\omega\|_{L^1}$ and we may take $s = 1$. Without loss of generality, we also assume that $\omega = e_1$. We decompose p_{e_1} into three pieces, when ξ is near zero, near e_1, and away from both. We write $p_{e_1} = p_1 + p_2 + p_3$ respectively defined by smooth cut-offs and define $h_i = \mathcal{F}^{-1} p_i$.

First, we consider $p_1(\xi) = p_\omega(\xi)\chi(100\xi)$ and write

$$p_1(\xi) := \phi(\xi) + g(\xi), \quad \text{where} \quad \phi(\xi) = \chi(100\xi) \frac{1 - 2\xi_1}{|\xi - e_1|^{2\alpha}}. \tag{10}$$

Note that $\phi \in \mathcal{S}(\mathbb{R}^n)$, and

$$g(\xi) = \left[\frac{1 - 2\xi_1}{|\xi - e_1|^{2\alpha} - |\xi|^{2\alpha}} - \frac{1 - 2\xi_1}{|\xi - e_1|^{2\alpha}} \right] \chi(100\xi),$$

is easily seen to satisfy the hypotheses of Lemma 2.3. So that

$$|\mathcal{F}[\chi(100\cdot)p_\omega(\cdot)](x)| = |h_1(x)| \lesssim \langle x\rangle^{-n-2\alpha}.$$

Moreover, analogous bounds hold for $p_2(\xi) = \chi(100|\xi-\omega|)p_\omega(\xi)$ and hence h_2.

We define $\chi_3(\cdot) = 1-\chi(100\cdot)-\chi(100|\cdot-\omega|)$, and consider $p_3(\xi) = \chi_3(\xi)p_\omega(\xi)$. Here we note that

$$p_3(\xi) = \chi_3(\xi)\frac{1-2\xi_1}{(|\xi|^2+1-2\xi_1)^\alpha - |\xi|^{2\alpha}} = \chi_3(\xi)|\xi|^{2-2\alpha}\eta\Big(\frac{1-2\xi_1}{|\xi|^2}\Big),$$

where

$$\eta(z) = \frac{z}{(1+z)^\alpha - 1}\chi_{[-1+c,C]}(z).$$

Here $\chi_{[-1+c,C]}(z)$ is smooth cut-off to the interval $[-1+c,C]$ for some $c,C>0$. Note that, on the support of $\chi_3(\xi)$, we have $\frac{1-2\xi_1}{|\xi|^2}\in[-1+c,C]$. Since η is analytic and bounded in an open neighborhood of this interval in the complex plane (the singularity at $z=0$ is removable), η has bounded derivatives to arbitrary order.[1] Using the chain rule, we see that

$$|\nabla^N p_3(\xi)| \lesssim \langle\xi\rangle^{2-2\alpha-N}.$$

Therefore, using Lemma 2.4, we conclude that

$$h_3(x) = \mathcal{F}^{-1}(p_3)(x) = O\big(\min(|x|^{-n-1},|x|^{-n+2\alpha-2})\big),$$

which implies that $h_3\in L^1$ (since $\alpha>1$). This yields the claim for $j=0$.

For $j>0$, note that

$$\partial_s\mathcal{F}^{-1}p_3(sx) = x\cdot[\nabla\mathcal{F}^{-1}p_3](xs) = \frac{1}{s}\mathcal{F}^{-1}(\nabla\cdot\xi\, p_3(\xi))(xs).$$

Similarly, $(s\partial_s)^\ell\mathcal{F}^{-1}p_3(sx) = \mathcal{F}^{-1}((\nabla\cdot\xi)^\ell p_3(\xi))(xs)$. Therefore,

$$|\partial_s^j s^n\mathcal{F}^{-1}p_3(sx)| \lesssim \sum_{\ell=0}^{j} s^{n+\ell-j}s^{-\ell}|\mathcal{F}^{-1}((\nabla\cdot\xi)^\ell p_3(\xi))(xs)|.$$

The claim for $h_3=\mathcal{F}^{-1}p_3$ follows from this as above since $(\nabla\cdot\xi)^\ell p_3(\xi)$ satisfies the same bounds as $p_3(\xi)$.

We now turn to p_1, and the proof follows as above since $(\nabla\cdot\xi)^\ell g$ satisfies the same bounds as g. For p_2, we write

$$p_2(\xi) = \chi(100|\xi-e_1|)\frac{2\xi_1-1}{|\xi|^{2\alpha}} + g_2(\xi),$$

and

$$g_2(\xi) = \bigg[\frac{1-2\xi_1}{|\xi-e_1|^{2\alpha}-|\xi|^{2\alpha}} - \frac{2\xi_1-1}{|\xi|^{2\alpha}}\bigg]\chi(100|\xi-e_1|).$$

[1] Also note that $\eta\gtrsim 1$ on the interval $[-1+c,C]$, which implies that on the support of χ_3, $p_\omega(\xi)\approx|\xi|^{2-2\alpha}$. Observing that $p_w\approx 1$ on the support of $\chi_1+\chi_2$ implies that $p_\omega(\xi)\approx\langle\xi\rangle^{2-2\alpha}$.

Now, $(\nabla\cdot\xi)^\ell g_2(\xi)$ satisfies the hypotheses of Lemma 2.3 (centered at $\xi = e_1$ instead of zero) for $\ell \leq 2$ and $\gamma = 2\alpha - 2 > 0$.

Now, we consider $h_{k,\epsilon}$. Let $H_\omega(\epsilon, x) = \mathcal{F}^{-1}\Big(p_\omega e^{-\epsilon p_\omega}\Big)(x)$. We first consider

$$H_3(\epsilon, x) := \mathcal{F}^{-1}\Big(\chi_3 p_\omega e^{-\epsilon p_\omega}\Big)(x) = \mathcal{F}^{-1}\Big(p_3 e^{-\epsilon p_\omega}\Big)(x).$$

Using the bounds on the derivative of p_3 and noting that $\sup_{\alpha\geq 0}\alpha^N e^{-\alpha} \lesssim 1$ for any $N \geq 0$, and that $|\nabla^j p_\omega| \lesssim |\nabla^j p_3|$ on the support of χ_3, and that $0 \leq p_3 \leq p_\omega$, we conclude that

$$\big|\nabla_\xi^N [p_3(\xi) e^{-\epsilon p_\omega(\xi)}]\big| \lesssim \frac{1}{\langle\xi\rangle^{2\alpha-2+N}}, \qquad N = 0, 1, 2, \dots$$

Therefore we have

$$|H_3(\epsilon, x)| = |\mathcal{F}^{-1}(p_3 e^{-\epsilon p_\omega})(x)| \lesssim \min(|x|^{-n-1}, |x|^{-n+2\alpha-2}), \tag{11}$$

uniformly in $\epsilon > 0$. This yields the claim for $j = 0$ for the contribution of H_3 to $h_{k,\epsilon} = s^n H_\omega(\epsilon, sx)$.

We now turn to

$$H_1(\epsilon, x) := \mathcal{F}^{-1}\Big(\chi_1 p_\omega e^{-\epsilon p_\omega}\Big)(x) = \mathcal{F}^{-1}\Big(p_1 e^{-\epsilon \widetilde{p}_1}\Big)(x),$$

where $\widetilde{p}_1(\xi) = \chi(10\xi)p_\omega(\xi)$. Using (10), we have $p_1 = \phi + g$. Defining $\widetilde{\phi}, \widetilde{g}$ analogously, we have $0 \leq \phi \leq \widetilde{\phi}$ and $0 \leq g \leq \widetilde{g}$. So that

$$p_1 e^{-\epsilon p_\omega} = \phi e^{-\epsilon\widetilde{\phi}} + \phi e^{-\epsilon\widetilde{\phi}}(e^{-\epsilon\widetilde{g}} - 1) + g e^{-\epsilon\widetilde{g}} e^{-\epsilon\widetilde{\phi}}.$$

The last two summand satisfy the hypotheses of Lemma 2.3 while the first summand is in $\mathcal{S}(\mathbb{R}^n)$ with uniform in ϵ bounds on the derivatives. Therefore,

$$|H_1(\epsilon, x)| \lesssim \langle x\rangle^{-n-2\alpha}$$

uniformly in $\epsilon > 0$. A similar argument for p_2 yields the same bounds for $H_2(\epsilon, x)$. Therefore, we conclude that

$$\sup_{\epsilon\geq 0} |H_\omega(\epsilon, x)| \lesssim \min(|x|^{-n-1}, |x|^{-n+2\alpha-2}),$$

which yields the claim.

Similarly, note that

$$\big|\nabla_\xi^N [p_3(\xi)(e^{-\epsilon p_\omega(\xi)} - 1)]\big| \lesssim \frac{\epsilon}{\langle\xi\rangle^{4\alpha-4+N}}, \qquad N = 0, 1, 2, \dots.$$

This implies the a.e. and L^1 convergence of the contribution of H_3 in $h_{k,\epsilon}$ to h_k.

For H_1 we write

$$p_1(\xi)(e^{-\epsilon p_\omega(\xi)} - 1) = \phi(\xi) e^{-\epsilon\widetilde{\phi}(\xi)}\Big(e^{-\epsilon\widetilde{g}(\xi)} - 1\Big) + \phi(\xi)\Big(e^{-\epsilon\widetilde{\phi}(\xi)} - 1\Big) + g(\xi)\Big(e^{-\epsilon\widetilde{p}_1(x)} - 1\Big).$$

The first and third summands satisfy the hypotheses of Lemma 2.3 with an additional factor of ϵ. The second summand is in $\mathcal{S}(\mathbb{R}^n)$ with all derivatives bounded by ϵ. A similar argument applies for the contribution of H_2, hence $h_{k,\epsilon}$ converges a.e. and in L^1 to h_k.

For the jth derivative of $h_{k,\epsilon}$, by chain rule and scaling as above, it suffices to prove that the L^1 norms of $\sup_\epsilon \epsilon^{j_1}\partial_\epsilon^{j_1}(x\cdot\nabla_x)^{j_2}\mathcal{F}^{-1}[p_\omega e^{-\epsilon p_\omega}](x)$ are $\lesssim 1$ for $j_1, j_2 \geq 0$ with $j_1 + j_2 \leq 2$. Noting that $|\epsilon^{j_1}\partial_\epsilon^{j_1} e^{-\epsilon p_\omega}| = |(-\epsilon p_\omega)^{j_1} e^{-\epsilon p_\omega}| \lesssim e^{-\epsilon p_\omega/2}$, the arguments above remain valid. Convergence of the s derivatives of $h_{k,\epsilon}$ follow similarly. □

Using Lemma 2.2 and dominated convergence theorem, we conclude that for $f \in \mathcal{S}$ and for all $x \in \mathbb{R}^n$,

$$\lim_{\epsilon\to 0^+} T^\alpha_{k,\epsilon} f(x) = \frac{i}{2|k|^{2\alpha-1}} \int_0^\infty e^{-it|k|/2} \int_{\mathbb{R}^n} h_k(y) f(x-y+t\omega)\, dy\, dt := T^\alpha_k f(x).$$

Following the notation of [55], for $\epsilon > 0$, let

$$G_\epsilon f = \int_{\mathbb{R}^n} T^\alpha_{k,\epsilon} f(k,\cdot) dk, \qquad G_0 f = \int_{\mathbb{R}^n} T^\alpha_k f(k,\cdot) dk,$$

Note that

$$G_\epsilon f(x) = \int_{\mathbb{R}^n} \frac{i}{2|k|^{2\alpha-1}} \int_0^\infty \int_{\mathbb{R}^n} e^{-i|k|t/2} h_{k,\frac{\epsilon t}{2|k|^{2\alpha-1}}}(y) f(k, x-y+t\omega)\, dy\, dt\, dk. \tag{12}$$

Passing to polar coordinates, $k = s\omega$, and changing the order of integration, we have

$$G_\epsilon f(x) = \frac{i}{2} \int_{S^{n-1}} \int_0^\infty F_\epsilon(t,\omega,x)\, dt\, d\omega,$$

where

$$F_\epsilon(t,\omega,x) = \int_0^\infty e^{-ist/2} s^{n-2\alpha} h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}} * f(s\omega,\cdot)(x+t\omega)\, ds.$$

Also note that $G_0 f$ satisfies the same formula with F_0 replacing F_ϵ.

Lemma 2.5. *Let $\epsilon > 0$, $1 \leq p \leq \infty$, and $f(k,x) \in \mathcal{S}(\mathbb{R}^n_k, \mathcal{S}(\mathbb{R}^n_x))$. For all $n > 2\alpha + 1$, we have*

$$\|G_\epsilon f\|_{L^p} \leq C_{n,m} \int_{\mathbb{R}^n} \langle k\rangle^2 \sum_{j=0}^{2} \|D^j_k f(k,\cdot)\|_{L^p} \frac{dk}{|k|^{1+2\alpha}}.$$

For $2\alpha < n \leq 2\alpha + 1$, we have

$$\|G_\epsilon f\|_{L^p} \leq C_{n,m} \int_{\mathbb{R}^n} \langle k\rangle^{\frac{n}{2}-\alpha+1} \sum_{j=0}^{3} \|D^j_k f(k,\cdot)\|_{L^p} \frac{dk}{|k|^{\frac{n}{2}+\alpha}}.$$

Moreover, $G_\epsilon f \to G_0 f$ in L^p as $\epsilon \to 0^+$.

Proof. Note that

$$\big\|F_\epsilon(t,\omega,x)\big\|_{L^p_x} \lesssim \int_0^\infty s^{n-2\alpha} \|\sup_\epsilon h_{s\omega,\epsilon}\|_{L^1} \|f(s\omega,\cdot)\|_{L^p}\, ds \lesssim \int_0^\infty s^{n-2\alpha} \|f(s\omega,\cdot)\|_{L^p}\, ds,$$

which suffices for the integral in $0 < t \lesssim 1$. For $t \gg 1$ and $n > 2\alpha+1$, we integrate by parts twice in the s integral to obtain

$$|F_\epsilon(t,\omega,x)| \lesssim \frac{1}{t^2}\int_{\mathbb{R}^n}\int_0^\infty \big|\partial_s^2\big(s^{n-2\alpha}h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}}(y)f(s\omega,x-y+t\omega)\big)\big|\,ds\,dy.$$

Let $H_{s\omega}(y) = |\sup_{\epsilon>0,j=0,1,2} s^j\partial_s^j h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}}(y)|$. Using this we obtain the bound

$$\begin{aligned}|F_\epsilon(t,\omega,x)| &\lesssim \frac{1}{t^2}\int_{\mathbb{R}^n}\int_0^\infty \langle s\rangle^2 s^{n-2\alpha-2}H_{s\omega}(y)\sum_{j=0}^2\big|\partial_s^j f(s\omega,x-y+t\omega)\big|\,ds\,dy\\ &\lesssim \frac{1}{t^2}\int_{\mathbb{R}^n}\int_0^\infty H_{s\omega}(y)\langle s\rangle^2 s^{n-2\alpha-2}\sum_{j=0}^2\big|\partial_s^j f(s\omega,x-y+t\omega)\big|\,ds\,dy.\end{aligned}$$

By Lemma 2.2, $\|H_{s\omega}\|_{L^1}\lesssim 1$, therefore uniformly in t and ω, we have

$$\big\|F_\epsilon(t,\omega,x)\big\|_{L^p_x} \lesssim \frac{1}{\langle t\rangle^2}\int_0^\infty \langle s\rangle^2 s^{n-2\alpha-2}\sum_{j=0}^2\big\|\partial_s^j f(s\omega,\cdot)\big\|_{L^p}\,ds,$$

which implies the claim for $G_\epsilon f$ when $n > 2\alpha+1$. The convergence of $G_\epsilon f$ to $G_0 f$ in L^p also follows by applying the same argument with $h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}} - h_{s\omega}$ replacing $h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}}$ and using dominated convergence theorem.

We now consider the case $2\alpha < n \le 2\alpha+1$ and $t \gg 1$. After an integration by parts, we have

$$F_\epsilon(t,\omega,x) = -\frac{2i}{t}\int_0^\infty e^{-ist/2}\partial_s[s^{n-2\alpha}h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}} * f(s\omega,\cdot)(x+t\omega)]\,ds.$$

We cannot integrate by parts again to gain another power of t. Therefore we utilize the identity (with $K(s) = \partial_s[s^{n-2\alpha}h_{s\omega,\frac{\epsilon t}{2s^{2\alpha-1}}} * f(s\omega,\cdot)(x+t\omega)]$)

$$\int_0^\infty e^{-ist/2}K(s)ds = \frac{1}{2}\int_0^{2\pi/t} e^{-ist/2}K(s)ds + \frac{1}{2}\int_0^\infty e^{-i(s+2\pi/t)t/2}[K(s+2\pi/t)-K(s)]ds.$$

This implies that (with $\eta = \frac{n}{2}-\alpha \in (0,\frac{1}{2}]$)

$$\begin{aligned}&\Big\|\int_0^\infty e^{-ist/2}K(s)ds\Big\|_{L^p_x} \lesssim\\ &\quad\int_0^{2\pi/t}\|K(s)\|_{L^p_x}ds + \int_0^\infty (\|K(s+2\pi/t)\|_{L^p_x} + \|K(s)\|_{L^p_x})^{1-\eta}\Big(\int_s^{s+2\pi/t}\big\|\partial_\rho K(\rho)\big\|_{L^p_x}d\rho\Big)^\eta ds\\ &\qquad\lesssim t^{-1}\sup_{0<s<1}\|K(s)\|_{L^p_x} + t^{-\eta}\int_0^\infty \big[\sup_{s<\rho<s+1}\|K(\rho)\|_{L^p}\big]^{1-\eta}\big[\sup_{s<\rho<s+1}\|\partial_\rho K(\rho)\|_{L^p}\big]^\eta ds.\end{aligned}$$

Note that

$$\|K(\rho)\|_{L^p_x} \lesssim \langle\rho\rangle\rho^{n-2\alpha-1}\big(\|f(\rho\omega,\cdot)\|_{L^p} + \|\partial_\rho f(\rho\omega,\cdot)\|_{L^p}\big)$$

$$\big\|\partial_\rho K(\rho)\big\|_{L^p_x} \lesssim \langle\rho\rangle^2\rho^{n-2\alpha-2}\big(\|f(\rho\omega,\cdot)\|_{L^p} + \|\partial_\rho f(\rho\omega,\cdot)\|_{L^p} + \|\partial_\rho^2 f(\rho\omega,\cdot)\|_{L^p}\big).$$

Therefore, for $t \gg 1$

$$\Big\| \int_0^\infty e^{-ist/2} K(s) ds \Big\|_{L^p_x} \lesssim t^{-\eta} \int_0^\infty \langle s\rangle s^{n-2\alpha-1} (s^{-1}\langle s\rangle)^\eta \sup_{s<\rho<s+1} \sum_{j=0}^2 \|\partial_\rho^j f(\rho\omega,\cdot)\|_{L^p} ds.$$

Noting that, for $s < \rho < s+1$

$$\sum_{j=0}^2 \|\partial_\rho^j f(\rho\omega,\cdot)\|_{L^p} \le \sum_{j=0}^2 \|\partial_s^j f(s\omega,\cdot)\|_{L^p} + \int_s^{s+1} \sum_{j=0}^3 \|\partial_\rho^j f(\rho\omega,\cdot)\|_{L^p}\, d\rho,$$

and applying Fubini's theorem yield the claim bounding G_ϵ in L^p. Convergence in L^p follows similarly. □

We now return to the operator Z_J defined in (7). Taking limits as $\epsilon_j \to 0$ using the lemmas above and tracing the steps in pages 13 and 14 of [16], we bound Z_j defined in (7) as

$$\|Z_J f\|_{L^p} \lesssim \|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \|f\|_{L^p},$$

where

$$F = F(\omega_1, y_1, t_1, \dots, \omega_J, y_J, t_J)$$
$$:= \int_{(0,\infty)^J} \prod_{j=1}^J \big[s_j^{n-2\alpha} e^{-i\frac{s_j t_j}{2}} h_{s_j\omega_j}(y_j)\big] K_J(s_1\omega_1, \dots, s_J\omega_J)\, ds_J \cdots ds_1,$$

where $K_J(k_1, k_2, \dots, k_J) = \prod_{j=1}^J \widehat{V}(k_j - k_{j-1})$.

The following lemma finishes the proof of L^p boundedness of Z_J.

Lemma 2.6. *For $2\alpha < n < 4\alpha - 1$, we have*

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \le C^J \|\langle\cdot\rangle^{\frac{4\alpha+1-n}{2}+} V(\cdot)\|_{L^2}^J,$$

for $n = 4\alpha - 1 \in \mathbb{N}$, we have

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \le C^J \|\langle x\rangle^{1+} V\|_{H^{0+}}^J,$$

for $n > 4\alpha - 1$ and $\sigma > \frac{n-2\alpha}{n-1}$, we have

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \le C^J \|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|_{L^{\frac{n-1}{n-2\alpha}-}}^J.$$

Here C depends on n, α and the actual values of $\pm$ signs.

Proof. We write F as a sum of 2^J operators of the form (for each subset $\mathcal{J}$ of $\{1, 2, ..., J\}$)

$$F_{\mathcal{J}}(\omega_1, y_1, t_1, \dots, \omega_J, y_J, t_J) = F(\omega_1, y_1, t_1, \dots, \omega_J, y_J, t_J) \big[\prod_{j\in\mathcal{J}} \chi(y_j)\big] \big[\prod_{j\notin\mathcal{J}} \widetilde{\chi}(y_j)\big].$$

It suffices to prove that each $F_{\mathcal{J}}$ satisfies the claim.

Fix $r \geq 2$ and $\frac{1}{q}+\frac{1}{r}=1$. By Hausdorff-Young inequality, we have (with $L^p(\Omega)L^q(D)=L^P(\Omega,L^q(D))$)

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \int_{(S^{n-1}\times\mathbb{R}^n)^J} \Big[\int_{(0,\infty)^J} \big[\prod_{j=1}^J s_j^{n-2\alpha} h_{s_j\omega_j}(y_j)\big]^q \times |K_J(s_1\omega_1,\dots,s_J\omega_J)|^q ds_1\dots ds_J\Big]^{1/q} \big[\prod_{j\in\mathcal{J}}\chi(y_j)\big]\big[\prod_{j\notin\mathcal{J}}\widetilde{\chi}(y_j)\big] d\vec{y}d\vec{\omega}.$$

Note that, by (11) in the proof of Lemma 2.2 above (for $0<\delta\ll 1$)

$$|h_{s\omega}(y)| \lesssim s^n \min((s|y|)^{-n-\delta},(s|y|)^{-n+\delta}) \lesssim \chi(y)|y|^{-n+\delta}s^{\delta}+\widetilde{\chi}(y)|y|^{-n-\delta}s^{-\delta}.$$

Since $\chi(y)|y|^{-n+\delta}\in L^1$ and $\widetilde{\chi}(y)|y|^{-n-\delta}\in L^1$ for any $\delta>0$, we can bound the norm above by

$$\int_{(S^{n-1})^J} \Big[\int_{(0,\infty)^J} \big[\prod_{j\in\mathcal{J}}^J s_j^{(n-2\alpha+\delta)q}\big]\big[\prod_{j\notin\mathcal{J}}^J s_j^{(n-2\alpha-\delta)q}\big] |K_J(s_1\omega_1,\dots,s_J\omega_J)|^q d\vec{s}\Big]^{1/q} d\vec{\omega}.$$

By Holder in ω_j integrals we conclude that

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}} \big[\prod_{j\in\mathcal{J}}^J |k_j|^{(n-2\alpha+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}}^J |k_j|^{(n-2\alpha-\delta)q-n+1}\big]\times |K_J(k_1,\dots,k_J)|^q dk_1\dots dk_J\Big]^{1/q}. \tag{13}$$

Similarly, (here $\alpha_j=0$ or 1 independently)

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \int_{(S^{n-1}\times\mathbb{R}^n)^J} \Big[\int_{(0,\infty)^J} \Big|\partial_{s_1}^{\alpha_1}\dots\partial_{s_J}^{\alpha_J}\prod_{j=1}^J \big(s_j^{n-2\alpha}h_{s_j\omega_j}(y_j)\big)\times K_J(s_1\omega_1,\dots,s_J\omega_J)\Big|^q ds_1\dots ds_J\Big]^{1/q} \times \big[\prod_{j\in\mathcal{J}}\chi(y_j)\big]\big[\prod_{j\notin\mathcal{J}}\widetilde{\chi}(y_j)\big] d\vec{y}d\vec{\omega}.$$

Since $\partial_s h_{s\omega}$ satisfies the same bounds as $\frac{1}{s}h_{s\omega}$, proceeding as above, we obtain the estimate

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}} \big[\prod_{j\in\mathcal{J}} |k_j|^{(n-2\alpha+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}} |k_j|^{(n-2\alpha-\delta)q-n+1}\big]\times \Big|\prod_{j=1}^J(\nabla_{k_j}^{\alpha_j}+|k_j|^{-\alpha_j})K_J(k_1,\dots,k_J)\Big|^q dk_1\dots dk_J\Big]^{1/q}.$$

Using Hardy's inequality, this implies that

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}} \big[\prod_{j\in\mathcal{J}} |k_j|^{(n-2\alpha+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}} |k_j|^{(n-2\alpha-\delta)q-n+1}\big]\times \Big|\prod_{j=1}^J \nabla_{k_j}^{\alpha_j} K_J(k_1,\dots,k_J)\Big|^q dk_1\dots dk_J\Big]^{1/q}. \tag{14}$$

Let $2\alpha < n < 4\alpha - 1$. Applying (13) with $0 < \delta < \frac{4\alpha-1-n}{2}$ and $q = r = 2$, we obtain

$$\|F_{\mathcal{J}}\|^2_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)} \lesssim \int_{\mathbb{R}^{nJ}} \big[\prod_{j\in\mathcal{J}} |k_j|^{n-4\alpha+1+2\delta}\big]\big[\prod_{j\notin\mathcal{J}} |k_j|^{n-4\alpha+1-2\delta}\big] |K_J(k_1,\dots,k_J)|^2 d\vec{k}.$$

Note that by Hardy's inequality the integral in k_J is bounded by

$$\int ||D_{k_J}|^{\frac{4\alpha-1-n}{2}\pm\delta}\widehat{V}(k_{J-1}-k_J)|^2 dk_J \lesssim \|\langle\cdot\rangle^{\frac{4\alpha-1-n}{2}\pm\delta}V(\cdot)\|^2_{L^2} \lesssim \|\langle\cdot\rangle^{\frac{4\alpha-1-n}{2}+\delta}V(\cdot)\|^2_{L^2}.$$

Repeated application of this inequality yields

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{\frac{4\alpha-1-n}{2}+\delta}V(\cdot)\|^J_{L^2}.$$

Similarly, applying (14) with $r = q = 2$ and $0 < \delta \ll 1$ yield

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{2+\frac{4\alpha-1-n}{2}+\delta}V(\cdot)\|^J_{L^2}.$$

We lose two powers here as derivatives in k_{j-1} and k_j may both hit $\widehat{V}(k_{j-1}-k_j)$. Writing

$$\prod_{j=1}^J(1+|t_j|) = \sum_{\alpha_1,\dots,\alpha_J\in\{0,1\}} |t_1^{\alpha_1}\dots t_J^{\alpha_J}|,$$

these inequalities imply with that

$$\|\prod_{j=1}^J\langle t_j\rangle F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{2+\frac{4\alpha-1-n}{2}+\delta}V(\cdot)\|^J_{L^2},$$

which by multilinear complex interpolation leads to

$$\|\prod_{j=1}^J\langle t_j\rangle^{\frac{1}{2}+}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{1+\frac{4\alpha-1-n}{2}+\delta+}V(\cdot)\|^J_{L^2}.$$

This proves the claim for $n < 4\alpha - 1$ by Cauchy-Schwarz in t integrals.

For $n = 4\alpha - 1 \in \mathbb{N}$, with $q = 2-$, $r = 2+$, (13) implies

$$\|F_{\mathcal{J}}\|^{2-}_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{2+}(\mathbb{R}^J)} \lesssim \int_{\mathbb{R}^{nJ}} \big[\prod_{j\notin\mathcal{J}}^J |k_j|^{0-}\big] |K_J(k_1,\dots,k_J)|^{2-} dk_1\dots dk_J.$$

By Hardy's inequality, the integral in k_J is

$$\begin{aligned}\lesssim \int \big||D_{k_J}|^{0+}\mathcal{F}(V(\cdot)e^{ik_{J-1}\cdot})(k_J)\big|^{2-}dk_J &\lesssim \int \big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot)e^{ik_{J-1}\cdot})(k_J)\big|^{2-}dk_J \\ \lesssim \int \big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))(k_J)\big|^{2-}dk_J &\lesssim \Big[\int \langle k_J\rangle^{0+}\big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))(k_J)\big|^2 dk_J\Big]^{\frac{2-}{2}} \lesssim \|\langle\cdot\rangle^{0+}V(\cdot)\|^{2-}_{H^{0+}}.\end{aligned}$$

Repeating the same argument in the remaining variables yield

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{2+}(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{0+}V(\cdot)\|^J_{H^{0+}}.$$

Similar modifications in the other inequalities imply the claim in this case.

When $n > 4\alpha - 1$, we apply the inequalities with $0 < \delta \ll 1$ and $q = \frac{n-1-\delta}{n-2\alpha}$, $r = \frac{n-1-\delta}{2\alpha-1-\delta}$ to obtain

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}} \prod_{j\notin\mathcal{J}} |k_j|^{0-}|K_J(k_1,\dots,k_J)|^q dk_1\dots dk_J\Big]^{1/q} \lesssim \|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))\|_{L^q}^J.$$

Similarly, we obtain

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J} F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle\cdot\rangle^{2+}V(\cdot))\|_{L^q}^J,$$

which implies that

$$\Big\|\prod_{j=1}^J \langle t_j\rangle F_{\mathcal{J}}\Big\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{\frac{n-1-\delta}{2\alpha-1-\delta}}(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle x\rangle^{2+}V)\|_{L^{\frac{n-1-\delta}{n-2\alpha}}}^J.$$

Interpolating the two bounds we obtain (with $\sigma > \frac{n-2\alpha}{n-1-\delta}$)

$$\Big\|\prod_{j=1}^J \langle t_j\rangle^\sigma F_{\mathcal{J}}\Big\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{\frac{n-1-\delta}{2\alpha-1-\delta}}(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle x\rangle^{2\sigma}V)\|_{L^{\frac{n-1-\delta}{n-2\alpha}}}^J,$$

which implies the claim by Hölder's inequality in t integrals.

□

The statement in Theorem 2.1 follows by keeping track of the relationship between q, r, σ and δ in the proof above.

3. Low Energies

In this section we prove the low energy result, that for sufficiently large ℓ the tail of the Born series (5) extends to a bounded operator on $L^p(\mathbb{R}^n)$. It is convenient to use a change of variables to respresent $W_{low,\ell}$ as

$$\frac{\alpha}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2\alpha-1}(\mathcal{R}_0^+(\lambda^{2\alpha})V)^k\mathcal{R}_V^+(\lambda^{2\alpha})(V\mathcal{R}_0^+(\lambda^{2\alpha}))^k V[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})]\,d\lambda$$

We begin by using the symmetric resolvent identity on the perturbed resolvent $\mathcal{R}_V^+(\lambda^{2\alpha})$. With $v = |V|^{\frac{1}{2}}$, $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, we define $M^+(\lambda) = U + v\mathcal{R}_0^+(\lambda^{2\alpha})v$. Recall that M^+ is invertible on L^2 in a sufficiently small neighborhood of $\lambda = 0$ provided that zero is a regular point of the spectrum. Using the symmetric resolvent identity, one has

$$\mathcal{R}_V^+(\lambda^{2\alpha})V = \mathcal{R}_0^+(\lambda^{2\alpha})vM^+(\lambda)^{-1}v.$$

We select the cut-off χ to be supported in this neighborhood. Therefore, we have

$$W_{low,\ell} = \frac{\alpha}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2\alpha-1}\mathcal{R}_0^+(\lambda^{2\alpha})v\Gamma_k(\lambda)v[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})]\,d\lambda,$$

where $\Gamma_0(\lambda) := M^+(\lambda)^{-1}$ and for $\ell \geq 1$

$$\Gamma_\ell(\lambda) := Uv\mathcal{R}_0^+(\lambda^{2\alpha})\big(V\mathcal{R}_0^+(\lambda^{2\alpha})\big)^{\ell-1}vM^+(\lambda)^{-1}v\big(\mathcal{R}_0^+(\lambda^{2\alpha})V\big)^{\ell-1}\mathcal{R}_0^+(\lambda^{2\alpha})vU. \tag{15}$$

To state the main result of this section, we define an operator $T : L^2 \to L^2$ with integral kernel $T(x,y)$ to be absolutely bounded if the operator with kernel $|T(x,y)|$ is bounded on L^2.

Proposition 3.1. *Fix $n > 2\alpha > 2$ and $0 < \eta$. Let Γ be a λ dependent absolutely bounded operator. Let*

$$\widetilde{\Gamma}(x,y) := \sup_{0<\lambda<\lambda_0} \Big[|\Gamma(\lambda)(x,y)| + \sup_{1\leq k \leq \lceil \frac{n}{2} \rceil + 1} \big|\lambda^{k-\eta}\partial_\lambda^k \Gamma(\lambda)(x,y)\big|\Big].$$

For $2\alpha < n < 4\alpha$ assume that $\widetilde{\Gamma}$ is bounded on L^2, and for $n \geq 4\alpha$ assume that $\widetilde{\Gamma}$ satisfies

$$\widetilde{\Gamma}(x,y) \lesssim \langle x\rangle^{-\frac{n}{2}-}\langle y\rangle^{-\frac{n}{2}-}. \tag{16}$$

Then the operator with kernel

$$K(x,y) = \int_0^\infty \chi(\lambda)\lambda^{2\alpha-1}\big[\mathcal{R}_0^+(\lambda^{2\alpha})v\Gamma(\lambda)v[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})]\big](x,y)d\lambda \tag{17}$$

is bounded on L^p for $1 \leq p \leq \infty$ provided that $\beta > n$.

Note that boundedness of the contribution of the tail follows from this proposition and the following

Lemma 3.2. *Fix $n > 2\alpha \geq 2$. Assume that $|V(x)| \lesssim \langle x\rangle^{-\beta}$, where $\beta > n+4$ when n is odd and $\beta > n+3$ when n is even. Also assume that zero is a regular point of the spectrum of H. Then, for some $\eta > 0$, the operator $\Gamma_\ell(\lambda)$ defined in* (15) *satisfies the hypothesis of Proposition 3.1 for all ℓ when $2\alpha < n < 4\alpha$ and for all sufficiently large ℓ when $n \geq 4\alpha$.*

We prove Proposition 3.1 below, and provide the argument for Lemma 3.2 in Section 4. To prove these results we need the following representations of the free resolvent given in [15], which were inspired by Lemmas 3.2 and 6.2 in [16].

Proposition 3.3. *Fix $\alpha > 0$ and $n \in \mathbb{N}$ with $n > 2\alpha$. Then, we have the representations, with $r = |x-y|$,*

$$\mathcal{R}_0^+(\lambda^{2\alpha})(x,y) = \frac{e^{i\lambda r}}{r^{n-2\alpha}}F(\lambda r), \text{ and}$$

$$[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})](x,y) = \lambda^{n-2\alpha}\big[e^{i\lambda r}F_+(\lambda r) + e^{-i\lambda r}F_-(\lambda r)\big], \tag{18}$$

where, for all $0 \leq N \leq \frac{n+1+4\alpha}{2}$,

$$|\partial_\lambda^N F(\lambda r)| \lesssim \lambda^{-N}\langle\lambda r\rangle^{\frac{n+1}{2}-2\alpha}, \quad |\partial_\lambda^N F_\pm(\lambda r)| \lesssim \lambda^{-N}\langle \lambda r\rangle^{-\frac{n-1}{2}}. \tag{19}$$

Further, for all $1 \leq N \leq \frac{n+1+4\alpha}{2}$ we have

$$|\partial_\lambda^N F(\lambda r)| \lesssim \lambda^{-N}(\lambda r)^{\min(1, n-2\alpha, 2\alpha-)}\langle\lambda r\rangle^{\frac{n+1}{2}-2\alpha}, \quad |\partial_\lambda^N F_\pm(\lambda r)| \lesssim \lambda^{-N}(\lambda r)\langle\lambda r\rangle^{-\frac{n-1}{2}}, \tag{20}$$

which improves the estimate above for $\lambda r \lesssim 1$.

Proposition 3.4. *Fix $\alpha > \frac{1}{2}$ and $n > 2\alpha$. Assume that H has no embedded eigenvalues. Then when $\lambda \gtrsim 1$, we have*

$$\|\langle x\rangle^{-\frac{1}{2}-}\mathcal{R}_V(\lambda^{2\alpha})\langle y\rangle^{-\frac{1}{2}-}\|_{L^2\to L^2} \lesssim \lambda^{1-2\alpha},$$

provided that $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 1$. Further, for any $j \in \mathbb{N}$ we have

$$\|\langle x\rangle^{-j-\frac{1}{2}-}\partial_\lambda^j\mathcal{R}_V(\lambda^{2\alpha})\langle y\rangle^{-j-\frac{1}{2}-}\|_{L^2\to L^2} \lesssim \lambda^{1-2\alpha},$$

provided that $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 1 + 2j$.

We have the corollaries of Proposition 3.3 is

Corollary 3.5. *Under the conditions of Proposition 3.3 we have*

$$\sup_{0<\lambda<1} |\mathcal{R}_0(\lambda^{2\alpha})(x,y)| \lesssim |x-y|^{2\alpha-n} + |x-y|^{-\frac{n-1}{2}}.$$

Further, for sufficiently small $\eta > 0$, we have

$$\sup_{0<\lambda<1} |\lambda^{\max(0,N-\eta)}\partial_\lambda^N\mathcal{R}_0(\lambda^{2\alpha})(x,y)| \lesssim |x-y|^{2\alpha+\eta-n} + |x-y|^{N-\frac{n-1}{2}}, \;\; 1 \leq N \leq \frac{n+1+4\alpha}{2}.$$

Corollary 3.6. *Let $E(\lambda)(r) := \mathcal{R}_0^+(\lambda^{2\alpha})(r) - \mathcal{R}_0^+(0)(r)$. Then, for $\lambda r \ll 1$ and all sufficiently small $\eta > 0$, we have*

$$|\partial_\lambda^N E(\lambda)(r)| \lesssim \lambda^{\eta-N} r^{2\alpha-n+\eta}, \;\; 0 \leq N \leq \frac{n+1+4\alpha}{2}.$$

When $\lambda r \gtrsim 1$, we have

$$|E(\lambda)(r)| \lesssim r^{\frac{1-n}{2}}\lambda^{\frac{n+1}{2}-2\alpha} + r^{2\alpha-n}, \;\; \textit{and}$$
$$|\partial_\lambda^N E(\lambda)(r)| \lesssim r^{\frac{1-n}{2}+N}\lambda^{\frac{n+1}{2}-2\alpha}, \;\; 1 \leq N \leq \frac{n+1+4\alpha}{2}.$$

With these resolvent bounds, we now show that the low energy portion extends to a bounded operator on the full range $1 \leq p \leq \infty$. We say an operator K with integral kernel $K(x,y)$ is admissible if

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)|\,dy + \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)|\,dx < \infty.$$

By the Schur test, it follows that an operator with admissible kernel is bounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Using the representations in Proposition 3.3 with $r_1 = |x - z_1|$ and $r_2 := |z_2 - y|$ we see that $K(x,y)$ is the difference of

$$K_\pm(x,y) = \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)}{r_1^{n-2\alpha}} \int_0^\infty e^{i\lambda(r_1\pm r_2)}\chi(\lambda)\lambda^{n-1}\Gamma(\lambda)(z_1,z_2)F(\lambda r_1)F_\pm(\lambda r_2)d\lambda dz_1 dz_2. \tag{21}$$

We write

$$K(x,y) =: \sum_{j=1}^4 K_j(x,y),$$

where the integrand in K_1 is restricted to the set $r_1, r_2 \lesssim 1$, in K_2 to the set $r_1 \approx r_2 \gg 1$, in K_3 to the set $r_2 \gg \langle r_1 \rangle$, in K_4 to the set $r_1 \gg \langle r_2 \rangle$. We define $K_{j,\pm}$ analogously.

Using the bounds of Proposition 3.3 for $\lambda r \ll 1$, we bound the contribution of $|K_{1,\pm}(x,y)|$ by

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_1,r_2\lesssim 1}}{r_1^{n-2\alpha}} \widetilde{\Gamma}(z_1,z_2)dz_1dz_2.$$

Therefore

$$\int |K_{1,\pm}(x,y)|dx \lesssim \big\||\cdot|^{2\alpha-n}\big\|_{L^1(B(0,1))}\|v\|_{L^2}^2\|\widetilde{\Gamma}\|_{L^2\to L^2} \lesssim 1,$$

uniformly in y. Similarly, provided that $2\alpha < n < 4\alpha$,

$$\int |K_{1,\pm}(x,y)|dy \lesssim \|\widetilde{\Gamma}\|_{L^2\to L^2}\|v\|_{L^2}\|v(\cdot)|x-\cdot|^{2\alpha-n}\|_{L^2} \lesssim 1$$

holds uniformly in x. When $n \geq 4\alpha$, we use the decay bound (16) on $\widetilde{\Gamma}$ to obtain

$$\int |K_{1,\pm}(x,y)|dy \lesssim \int \langle z_1\rangle^{-n-}\langle z_2\rangle^{-n-} r_1^{2\alpha-n} dz_1dz_2 \lesssim 1,$$

which implies that K_1 is admissible.

For K_2, we restrict ourself to $K_{2,-}$ since the $+$ sign is easier to handle. We integrate by parts twice in the λ integral when $\lambda|r_1-r_2| \gtrsim 1$ (using Proposition 3.3 and the definition of $\widetilde{\Gamma}$) and estimate directly when $\lambda|r_1-r_2| \ll 1$ to obtain

$$\begin{aligned}|K_{2,-}(x,y)| &\lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\chi_{r_1\approx r_2\gg 1}}{r_1^{n-2\alpha}} \int_0^\infty \chi(\lambda)\lambda^{n-1}\chi(\lambda|r_1-r_2|)\langle \lambda r_1\rangle^{1-2\alpha} d\lambda dz_1dz_2 \\ &+ \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\chi_{r_1\approx r_2\gg 1}}{r_1^{n-2\alpha}} \int_0^\infty \frac{\chi(\lambda)\lambda^{n-3}\widetilde{\chi}(\lambda|r_1-r_2|)\langle \lambda r_1\rangle^{1-2\alpha}}{|r_1-r_2|^2} d\lambda dz_1dz_2 \\ &\lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\chi_{r_1\approx r_2\gg 1}}{r_1^{n-2\alpha}} \int_0^\infty \frac{\chi(\lambda)\lambda^{n-1}\langle \lambda r_1\rangle^{1-2\alpha}}{\langle \lambda(r_1-r_2)\rangle^2} d\lambda dz_1dz_2.\end{aligned}$$

Therefore, passing to the polar coordinates in x integral (centered at z_1) and noting $1-2\alpha < 0$, we have

$$\begin{aligned}\int |K_{2,-}(x,y)|dx &\lesssim \int_{\mathbb{R}^{2n}} \int_0^1 \int_{r_1\approx r_2\gg 1} v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\frac{\lambda^{n-2\alpha}}{\langle \lambda(r_1-r_2)\rangle^2} dr_1 d\lambda dz_1dz_2 \\ &\lesssim \int_{\mathbb{R}^{2n}} \int_0^1 \int_{\mathbb{R}} v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\frac{\lambda^{n-2\alpha-1}}{\langle \eta\rangle^2} d\eta d\lambda dz_1dz_2 \lesssim 1,\end{aligned}$$

uniformly in y. In the second line we defined $\eta = \lambda(r_1-r_2)$ in the r_1 integral and used $n-2\alpha-1 > -1$. Since $r_1 \approx r_2$, the integral in y can be bounded uniformly in x and hence the contribution of K_2 is admissible. We now consider the contribution of

$$K_{4,\pm}(x,y) = \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_1\gg\langle r_2\rangle}}{r_1^{n-2\alpha}} \int_0^\infty e^{i\lambda(r_1\pm r_2)}F(\lambda r_1)\chi(\lambda)\Gamma(\lambda)(z_1,z_2)\lambda^{n-1}F_\pm(\lambda r_2)\, d\lambda dz_1dz_2. \tag{22}$$

When $\lambda r_1 \lesssim 1$, using (19), we bound $|F_\pm(\lambda r_2)|, |F(\lambda r_1)| \lesssim 1$ and estimate the λ integral by $r_1^{-n}\widetilde{\Gamma}(z_1,z_2)$, whose contribution to K_4 is bounded by

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\widetilde{\Gamma}(z_1,z_2)\chi_{r_1\gg\langle r_2\rangle}}{r_1^{2n-2\alpha}} dz_1 dz_2.$$

This is admissible since $n > 2\alpha$.

When $\lambda r_1 \gtrsim 1$, we integrate by parts $N = \lceil n/2 \rceil + 1$ times (using (19)) to obtain the bound

$$\frac{1}{|r_1 \pm r_2|^N} \int_0^\infty \Big|\partial_\lambda^N \big[F(\lambda r_1)\widetilde{\chi}(\lambda r_1)\chi(\lambda)\lambda^{n-1}\Gamma(\lambda)(z_1,z_2)F_\pm(\lambda r_2)\big]\Big| d\lambda$$

$$\lesssim r_1^{-N} \sum_{0\le j_1+j_2+j_3+j_4\le N,\, j_i\ge 0} \int_{\frac{1}{r_1}}^1 \lambda^{\frac{n+1}{2}-2\alpha-j_1} r_1^{\frac{n+1}{2}-2\alpha} \lambda^{n-1-j_2} \big|\partial_\lambda^{j_3}\Gamma(\lambda)(z_1,z_2)\big| \frac{\lambda^{-j_4}}{\langle \lambda r_2\rangle^{\frac{n-1}{2}}} d\lambda$$

$$\lesssim r_1^{\frac{n+1}{2}-2\alpha-N}\widetilde{\Gamma}(z_1,z_2) \sum_{0\le j_1+j_2+j_3+j_4\le N,\, j_i\ge 0} \int_{\frac{1}{r_1}}^1 \lambda^{\frac{3n-1}{2}-2\alpha-j_1-j_2-j_3-j_4} d\lambda$$

$$\lesssim r_1^{-2\alpha-\frac{1}{2}-\{\frac{n}{2}\}}\widetilde{\Gamma}(z_1,z_2) \int_{\frac{1}{r_1}}^1 \lambda^{n-2\alpha-\frac{3}{2}-\{\frac{n}{2}\}} d\lambda \lesssim r_1^{-2\alpha-\epsilon}\widetilde{\Gamma}(z_1,z_2) \int_{\frac{1}{r_1}}^1 \lambda^{n-2\alpha-1-\epsilon} d\lambda$$

$$\lesssim r_1^{-2\alpha-\epsilon}\widetilde{\Gamma}(z_1,z_2),$$

provided that $0 < \epsilon < n - 2\alpha$. The contribution of this to (22) is

$$\lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_1\gg\langle r_2\rangle}}{r_1^{n+\epsilon}} dz_1 dz_2,$$

which is admissible as $\epsilon > 0$.

We now consider K_3:

$$K_3(x,y) = \int_{\mathbb{R}^{2n}} \chi_{r_2\gg\langle r_1\rangle} v(z_1)v(z_2) \int_0^\infty \lambda^{2\alpha-1}\chi(\lambda)\mathcal{R}_0^+(\lambda^{2\alpha})(r_1)\Gamma(\lambda)(z_1,z_2)[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})](r_2)\, d\lambda dz_1 dz_2. \tag{23}$$

We write

$$\mathcal{R}_0^+(\lambda^{2\alpha}) = \mathcal{R}_0^+(0) + [\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^+(0)] =: G_0 + E(\lambda),$$

$$\Gamma(\lambda) = \Gamma(0) + [\Gamma(\lambda) - \Gamma(0)] =: \Gamma(0) + \Gamma_1(\lambda).$$

Here $G_0 = \mathcal{R}_0^+(0) = c_{n,\alpha} r_1^{2\alpha-n}$. We first consider the contribution of $G_0\Gamma(0)$ to K_3:

$$\int_{\mathbb{R}^{2n}} \chi_{r_2\gg\langle r_1\rangle} v(z_1)v(z_2)G_0(r_1)\Gamma(0)(z_1,z_2) \int_0^\infty \lambda^{2\alpha-1}\chi(\lambda)[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^-(\lambda^{2\alpha})](r_2)\, d\lambda dz_1 dz_2.$$

Identifying the λ integral as a constant multiple of the kernel of $\chi((-\Delta)^{\frac{1}{2\alpha}})$, we may bound it as $O(\langle r_2\rangle^{-N})$ for all N since $\chi(|\xi|^{\frac{1}{\alpha}})$ is Schwartz. Therefore, we have the bound

$$\int_{\mathbb{R}^{2n}} \chi_{r_2\gg\langle r_1\rangle} v(z_1)v(z_2) r_1^{2\alpha-n} r_2^{-n-1}\widetilde{\Gamma}(z_1,z_2) dz_1 dz_2,$$

which is admissible by Lemma 4.1.

It remains to consider the contributions of $\mathcal{R}_0^+(\lambda^{2\alpha})\Gamma_1(\lambda)$ and of $E(\lambda)\Gamma(0)$. The former can be written as

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}}{r_1^{n-2\alpha}} \int_0^\infty e^{i\lambda(r_1\pm r_2)} F(\lambda r_1)\chi(\lambda)\lambda^{n-1}\Gamma_1(\lambda)(z_1,z_2)F_\pm(\lambda r_2)\, d\lambda dz_1 dz_2.$$

When $\lambda r_2 \ll 1$, using $|\Gamma_1(\lambda)| \lesssim \lambda^\eta\widetilde{\Gamma}$, which follows from the mean value theorem, and Proposition 3.3 to directly integrate in λ, we obtain the bound

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}}{r_1^{n-2\alpha} r_2^{n+\eta}}\widetilde{\Gamma}(z_1,z_2)dz_1dz_2,$$

which is admissible by Lemma 4.1 as $\eta > 0$. When $\lambda r_2 \gtrsim 1$, integrating by parts $N = \lceil n/2\rceil + 1$ times, we have the bound

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}}{r_1^{n-2\alpha}|r_1\pm r_2|^N} \int_0^\infty \Big|\partial_\lambda^N\big[F(\lambda r_1)\chi(\lambda)\widetilde{\chi}(\lambda r_2)\lambda^{n-1}\Gamma_1(\lambda)(z_1,z_2)F_\pm(\lambda r_2)\big]\Big|\, d\lambda dz_1dz_2. \tag{24}$$

We estimate the λ integral by (noting that $\lambda^{j_3}|\partial_\lambda^{j_3}\Gamma_1| \lesssim \lambda^\eta\widetilde{\Gamma}$ and using Proposition 3.3)

$$\begin{aligned}
&\lesssim r_2^{-\frac{n-1}{2}}\widetilde{\Gamma}(z_1,z_2) \sum_{0\le j_1+j_2+j_3+j_4\le N,\, j_i\ge 0} \int_{\frac{1}{r_2}}^1 \langle\lambda r_1\rangle^{\frac{n+1}{2}-2\alpha}\lambda^{-j_1}\lambda^{n-1-j_2}\lambda^{\eta-j_3}\lambda^{-\frac{n-1}{2}-j_4}d\lambda \\
&\lesssim r_2^{-\frac{n-1}{2}}\widetilde{\Gamma}(z_1,z_2)\int_{\frac{1}{r_2}}^1 \langle\lambda r_1\rangle^{\frac{n+1}{2}-2\alpha}\lambda^{\frac{n+1}{2}-\lceil\frac{n}{2}\rceil+\eta-2}d\lambda \\
&\lesssim r_2^{-\frac{n-1}{2}}\widetilde{\Gamma}(z_1,z_2)\Big(\int_{\frac{1}{r_2}}^{\min(\frac{1}{r_1},1)}\lambda^{\eta-\{\frac{n}{2}\}-\frac{3}{2}}d\lambda + \int_{\min(\frac{1}{r_1},1)}^1 r_1^{\frac{n+1}{2}-2\alpha}\lambda^{\frac{n}{2}-2\alpha+\eta-\{\frac{n}{2}\}-1}d\lambda\Big).
\end{aligned}$$

For $0 < \eta \ll 1$, the first integral is at most $r_2^{-\eta+\{\frac{n}{2}\}+\frac{1}{2}}$. Its contribution to (24) is at most

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}}{r_1^{n-2\alpha}r_2^{n+\eta}}\widetilde{\Gamma}(z_1,z_2)dz_1dz_2,$$

which is admissible by Lemma 4.1. Similarly, the second integral is bounded by $r_1^{n-2\alpha}r_1^{\frac{1}{2}+\{\frac{n}{2}\}}$ after multiplying the integrand by $(\lambda r_1)^{\frac{n-1}{2}}$ (assuming that $\eta \le n-2\alpha$). Contribution of this to (24) is at most

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}\langle r_1\rangle^{\frac{1}{2}+\{\frac{n}{2}\}}}{r_2^{n+\frac{1}{2}+\{\frac{n}{2}\}}}\widetilde{\Gamma}(z_1,z_2)dz_1dz_2,$$

which, by Lemma 4.1, is also admissible.

We now consider the contribution of $E(\lambda)\Gamma(0)$:

$$\int_{\mathbb{R}^{2n}} v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}\Gamma(0)(z_1,z_2)\int_0^\infty e^{\pm i\lambda r_2}E(\lambda)(r_1)\chi(\lambda)\lambda^{n-1}F_\pm(\lambda r_2)\, d\lambda dz_1dz_2. \tag{25}$$

Using Proposition 3.3 and Corollary 3.6 when $\lambda r_1 \ll 1$. Using this when $\lambda r_2 \ll 1$ and using $|\Gamma(0)(z_1,z_2)| \le \widetilde{\Gamma}(z_1,z_2)$, we bound (25) by direct estimate by

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2\gg\langle r_1\rangle}\widetilde{\Gamma}(z_1,z_2)}{r_1^{n-2\alpha-\eta}r_2^{n+\eta}}dz_1dz_2,$$

which is admissible by Lemma 4.1 since $\eta, n-2\alpha > 0$.

When $\lambda r_2 \gtrsim 1$ and $\lambda r_1 \ll 1$, we integrate by parts $N = \lceil n/2 \rceil + 1$ times to obtain

$$\int_{\mathbb{R}^{2n}} r_2^{-N} v(z_1)v(z_2)\chi_{r_2 \gg \langle r_1 \rangle}\widetilde{\Gamma}(z_1,z_2) \int_0^\infty \left| \partial_\lambda^N \left[E(\lambda)(r_1)\chi(\lambda)\chi(\lambda r_1)\widetilde{\chi}(\lambda r_2)\lambda^{n-1}F_\pm(\lambda r_2)\right]\right| \, d\lambda dz_1 dz_2.$$

Using Corollary 3.6 and Proposition 3.3, we bound this by

$$\int_{\mathbb{R}^{2n}} r_2^{-N+\frac{1-n}{2}} r_1^{\eta+2\alpha-n} v(z_1)v(z_2)\chi_{r_2 \gg \langle r_1 \rangle}\widetilde{\Gamma}(z_1,z_2) \int_{r_2^{-1}}^1 \lambda^{\eta - \{\frac{n}{2}\} - \frac{3}{2}} \, d\lambda dz_1 dz_2$$

$$\lesssim \int_{\mathbb{R}^{2n}} r_2^{-n-\eta} r_1^{\eta+2\alpha-n} v(z_1)v(z_2)\chi_{r_2 \gg \langle r_1 \rangle}\widetilde{\Gamma}(z_1,z_2) dz_1 dz_2,$$

which is again admissible by Lemma 4.1.

It remains to consider the case $\lambda r_1 \gtrsim 1$. Integrating by parts once we rewrite the λ integral in (25) as

$$\frac{1}{r_2}\int_0^\infty e^{\pm i\lambda r_2} \partial_\lambda [E(\lambda)(r_1)] \; \widetilde{\chi}(\lambda r_1)\chi(\lambda)\lambda^{n-1}F_\pm(\lambda r_2) \, d\lambda \tag{26}$$

$$+\frac{1}{r_2}\int_0^\infty e^{\pm i\lambda r_2} E(\lambda)(r_1) \; \partial_\lambda \left[\widetilde{\chi}(\lambda r_1)\chi(\lambda)\lambda^{n-1}F_\pm(\lambda r_2)\right] \, d\lambda. \tag{27}$$

For the second integral, (27), we integrate by parts $N = \lceil \frac{n}{2} \rceil$ more times using Proposition 3.3, to obtain the bound

$$\frac{1}{r_2^{N+\frac{n}{2}+\frac{1}{2}}} \sum_{j_1+j_2 \le N, \; 0 \le j_1, j_2} \int_{r_1^{-1}}^1 \left|\partial_\lambda^{j_1}[E(\lambda)(r_1)]\right| \lambda^{\frac{n-3}{2}-j_2} \, d\lambda.$$

Using Corollary 3.6 we bound this by

$$\lesssim \frac{1}{r_2^{N+\frac{n}{2}+\frac{1}{2}}}\Big[\int_{r_1^{-1}}^1 r_1^{2\alpha-n}\lambda^{\frac{n-3}{2}-N} \, d\lambda + \sum_{j_1+j_2 \le N, \; 0 \le j_1, j_2} \int_{r_1^{-1}}^1 r_1^{j_1+\frac{1-n}{2}} \lambda^{n-2\alpha-1-j_2} \, d\lambda\Big].$$

The first integral takes care of the additional term that arises in Corollary 3.6 (for $\lambda r \gtrsim 1$) in the case $j_1 = 0$. Letting $\{n/2\} = n/2 - \lfloor n/2 \rfloor$, we bound this by

$$\lesssim \frac{r_1^{\{n/2\}+\frac{1}{2}+2\alpha-n} + r_1^{\{n/2\}+\frac{1}{2}}}{r_2^{n+\{n/2\}+\frac{1}{2}}} \lesssim \frac{r_1^{\{n/2\}+\frac{1}{2}}}{r_2^{n+\{n/2\}+\frac{1}{2}}},$$

whose contribution is admissible by Lemma 4.2 since $r_2 \gg \langle r_1 \rangle$.

For the first integral, (26), we integrate by parts $N = \lceil \frac{n}{2} \rceil$ more times after pulling out the phase $e^{i\lambda r_1}$ to obtain the bound

$$\frac{1}{r_2^{\frac{n}{2}+\frac{1}{2}}|r_1 \pm r_2|^N} \sum_{j_1+j_2 \le N, \; 0 \le j_1, j_2} \int_{r_1^{-1}}^1 \left|\partial_\lambda^{j_1}[\widetilde{E}(\lambda)(r_1)]\right| \lambda^{\frac{n-1}{2}-j_2} \, d\lambda$$

$$\widetilde{E}(\lambda)(r_1) := e^{-i\lambda r_1}\partial_\lambda[E(\lambda)(r_1)]$$

Using the representation in Proposition 3.3, we bound this by

$$\frac{1}{r_2^{n+\{n/2\}+\frac{1}{2}}} \sum_{j_1+j_2 \le N, \; 0 \le j_1, j_2} \int_{r_1^{-1}}^1 r_1 \frac{(\lambda r_1)^{\frac{n+1}{2}-2\alpha}}{r_1^{n-2\alpha}} \lambda^{\frac{n-1}{2}-j_1-j_2} \, d\lambda$$

$$\lesssim \frac{1}{r_2^{n+\{n/2\}+\frac{1}{2}} r_1^{\frac{n-3}{2}}} \int_{r_1^{-1}}^{1} \lambda^{\frac{n}{2}-2\alpha-\{n/2\}}\, d\lambda \lesssim \frac{r_1^{\{n/2\}+\frac{1}{2}}}{r_2^{n+\{n/2\}+\frac{1}{2}}},$$

which is admissible by Lemma 4.2. □

4. Proof of Lemma 3.2

For small energies, it remains only to prove Lemma 3.2 stating that the operators $\Gamma_\ell(\lambda)$ defined in (15) satisfy the bounds needed to apply Proposition 3.1. This follows, with some modifications, from the discussion preceeding Lemma 3.5 in [16], also see Section 4 of [17]. We briefly sketch the argument here. One of the differences here is that $n-2\alpha$ can be close to zero, in which case we cannot gain a full power of λ in the bound (28) below.

We write $n_\star$ to denote $n+4$ if n is odd and $n+3$ if n is even. By Corollary 3.5, for a sufficiently small $\eta>0$, the operator R_j with kernel

$$R_j(x,y) := v(x)v(y) \sup_{0<\lambda<1} |\lambda^{\max(0,j-\eta)} \partial_\lambda^j \mathcal{R}_0^+(\lambda^{2\alpha})(x,y)| \tag{28}$$

is bounded on $L^2(\mathbb{R}^n)$ for $0 \le j \le \lceil \frac{n}{2} \rceil + 1$ provided that $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n_\star$. This follows the argument from Section 4 of [17].

Similarly by Corollary 3.6, $\mathcal{E}(\lambda) := v[\mathcal{R}_0^+(\lambda^{2\alpha}) - \mathcal{R}_0^+(0)]v$ satisfies

$$\|\mathcal{E}(\lambda)\|_{L^2\to L^2} \lesssim \lambda^\eta.$$

Now, we define the operator

$$T_0 := U + v\mathcal{R}_0^+(0)v = M^+(0).$$

By the assumption that zero energy is regular, T_0 is invertible with absolutely bounded inverse, see [15]. Note that by a Neumann series expansion and the invertibility of T_0 we have

$$[M^+(\lambda)]^{-1} = \sum_{k=0}^{\infty} (-1)^k T_0^{-1} (\mathcal{E}(\lambda) T_0^{-1})^k.$$

The series converges in the operator norm on L^2 for sufficiently small λ. By the resolvent identity the operator $\partial_\lambda^N [M^+(\lambda)]^{-1}$ is a linear combination of operators of the form

$$[M^+(\lambda)]^{-1} \prod_{j=1}^{J} \big[v\big(\partial_\lambda^{N_j} \mathcal{R}_0^+(\lambda^{2\alpha})\big) v [M^+(\lambda)]^{-1}\big],$$

where $\sum N_j = N$ and each $N_j \ge 1$. From the discussion above on R_j's this representation implies that

$$\sup_{0<\lambda<\lambda_0} \lambda^{\max(0,N-\eta)} |\partial_\lambda^N [M^+(\lambda)]^{-1}(x,y)| \tag{29}$$

is bounded in L^2 for $N = 0,1,\dots,\lceil \frac{n}{2} \rceil + 1$ provided that $\beta > n_\star$.

Recalling the definition of $\Gamma_\ell(\lambda)$, (15), and noting the L^2 boundedness of R_j's above we see that

$$\sup_{0<\lambda<\lambda_0} \lambda^{\max(0,N-\eta)} \big|\partial_\lambda^N \big(Uv\mathcal{R}_0^+(\lambda^{2\alpha})\big(V\mathcal{R}_0^+(\lambda^{2\alpha})\big)^{\ell-1} v\big)(x,y)\big|$$

is bounded on L^2. This yields Lemma 3.2 when $2\alpha < n < 4\alpha$.

For $n \geq 4\alpha$, Lemma 3.2 follows by writing

$$\Gamma_k(\lambda) = UvA(\lambda)vM^{-1}(\lambda)vA(\lambda)vU,$$

where

$$A(\lambda, z_1, z_2) = \big[\big(\mathcal{R}_0^+(\lambda^{2\alpha})V\big)^{\ell-1}\mathcal{R}_0^+(\lambda^{2\alpha})\big](z_1, z_2). \tag{30}$$

If $\ell - 1$ is sufficiently large depending on n, α and $|V(x)| \lesssim \langle x\rangle^{-\frac{n_\star}{2}-}$, then

$$\sup_{0<\lambda<1} |\lambda^{\max\{0,k-\eta\}} \partial_\lambda^k A(\lambda, z_1, z_2)| \lesssim \langle z_1\rangle^{\frac{3}{2}+\{\frac{n}{2}\}} \langle z_2\rangle^{\frac{3}{2}+\{\frac{n}{2}\}},$$

for $0 \leq \ell \leq \lceil \frac{n}{2} \rceil + 1$. This follows from the pointwise bounds on R_j above. The iteration of the resolvents smooths out the local singularity $|x - \cdot|^{2\alpha-n}$. Each iteration improves the local singularity by 2α, so that after j iterations the local singularity is of size $|x-\cdot|^{2\alpha j-n}$. Selecting $\ell-1$ large enough ensures that the local singularity is completely integrated away. This yields Lemma 3.2, see [16, 15] for more details.

We recall Lemmas 4.1 and 4.2 from [17], which we used in the proof of Proposition 3.1, which we state here for the convenience of the reader.

Lemma 4.1. *Let K be an operator with integral kernel $K(x, y)$ that satisfies the bound*

$$|K(x, y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\widetilde{\Gamma}(z_1, z_2)\chi_{\{|y-z_2|\gg\langle z_1-x\rangle\}}}{|x - z_1|^{n-2\alpha-k}|z_2 - y|^{n+\ell}}\, dz_1\, dz_2$$

for some $0 \leq k \leq n - 2\alpha$ and $\ell > 0$. Then, under the hypotheses of Lemma 3.1, the kernel of K is admissible, and consequently K is a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Lemma 4.2. *Let K be an operator with integral kernel $K(x, y)$ that satisfies the bound*

$$|K(x, y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\widetilde{\Gamma}(z_1, z_2)\chi_{\{|y-z_2|\gg\langle z_1-x\rangle\}}|x - z_1|^\ell}{|z_2 - y|^{n+\ell}}\, dz_1\, dz_2$$

for some $\ell > 0$. Then, under the hypotheses of Lemma 3.1, the kernel of K is admissible, and consequently K is a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

5. High Energy

Since we can control the contribution of the Born series to arbitrary length, and the low energy portion of the tail of the series in (5), we need only consider the tail when $\lambda \gtrsim 1$ and show that

$$\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2\alpha-1}[(\mathcal{R}_0^+V)^\ell V\mathcal{R}_V^+(V\mathcal{R}_0^+)^\ell V\mathcal{R}_0^\pm](\lambda^{2\alpha})\,d\lambda$$

extends to bounded operators on $L^p(\mathbb{R}^n)$ provided ℓ is sufficiently large. In all cases we assume there are no positive eigenvalues of H. The argument follows is identical to that [16] for the integer order Schrödinger case using the bounds on the resolvents in Proposition 3.3 and the limiting absorption principle in Proposition 3.4, which are tailored to the fractional Schrödinger operators. This allows us to pointwise dominate the integral kernel of the tail of the Born series with an admissible kernel:

Proposition 5.1. *We have the bound*

$$\left|\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2\alpha-1}(\mathcal{R}_0^+(\lambda^{2\alpha})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2\alpha})V(\mathcal{R}_0^+(\lambda^{2\alpha})V)^\ell\mathcal{R}_0^\pm(\lambda^{2\alpha})(x,y)\,d\lambda\right| \lesssim \frac{1}{\langle|x|-|y|\rangle^{\frac{n+3}{2}}\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}, \tag{31}$$

provided ℓ is sufficiently large, and $|V(x)|\lesssim\langle x\rangle^{-\beta}$ for some $\beta>n_\star$.

The proof is a straightforward modification of the proof of Propositions 5.3 and 6.5 in [16]. To complete the claim, we use Lemma 5.2 from [16] (also see Lemma 3.1 in [59] and Lemma 2.1 in [27]):

Lemma 5.2. *Suppose that K is an integral operator whose kernel obeys the pointwise bounds*

$$|K(x,y)|\lesssim\frac{1}{\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}\langle|x|-|y|\rangle^{\frac{n+1}{2}+\epsilon}}. \tag{32}$$

Then K is admissible provided that $\epsilon>0$.

The proposition and lemma imply that the kernel is admissible and hence the tail extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1\le p\le\infty$.

References

[1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

[2] M. Beceanu, *Structure of wave operators for a scaling-critical class of potentials.* Amer. J. Math. 136 (2014), no. 2, 255–308.

[3] M. Beceanu, and W. Schlag, *Structure formulas for wave operators.* Amer. J. Math. 142 (2020), no. 3, 751–807.

[4] M. Beceanu, and W. Schlag, *Structure formulas for wave operators under a small scaling invariant condition.* J. Spectr. Theory 9 (2019), no. 3, 967–990.

[5] J. Bellazzini, V. S. Georgiev and N. Visciglia, *Long time dynamics for semi-relativistic NLS and half wave in arbitrary dimension*, Math. Ann. **371** (2018), no. 1-2, 707–740

[6] H. Cheng, A. Soffer, Z. Wu, X. Yao, *The L^p-boundedness of wave operators for higher order Schrödinger operator with zero singularities in low odd dimensions*, preprint, 2025, 57pp. arXiv:2505.07009

[7] Y. Cho, H. Hajaiej, H. Hwang, and T. Ozawa. *On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity*, Funkcial. Ekvac. **56** (2013), no. 2, 193–224

[8] Y. Cho and S. H. Lee, *Strichartz estimates in spherical coordinates*, Indiana Univ. Math. J. **62** (2013), no. 3, 991–1020.

[9] Y. Cho, T. Ozawa and S. Xia, *Remarks on some dispersive estimates, Commun. Pure Appl. Anal.* **10** (2011), no. 4, 1121–1128.

[10] J.-C. Cuenin, *Embedded eigenvalues of generalized Schrödinger operators*, J. Spectr. Theory **10** (2020), no. 2, 415–437.

[11] P. D'Ancona, and L. Fanelli, *Lp-boundedness of the wave operator for the one dimensional Schrödinger operator*, Commun. Math. Phys. 268 (2006), 415–438.

[12] V. D. Dinh, *A study on blowup solutions to the focusing L^2-supercritical nonlinear fractional Schrödinger equation*, J. Math. Phys. **59** (2018), no. 7, 071506, 25 pp.

[13] M. B. Erdoğan, M. J. Goldberg and W. R. Green, *Counterexamples to L^p boundedness of wave operators for classical and higher order Schrödinger operators*, J. Funct. Anal. **285** (2023), no. 5, Paper No. 110008, 18 pp.

[14] M. B. Erdoğan, M. J. Goldberg and W. R. Green, *Dispersive estimates for higher order Schrödinger operators with scaling-critical potentials*, Math. Ann. **392** (2025), no. 2, 2225–2252.

[15] M. B. Erdoğan, M. Goldberg, and W. R. Green, *dispersive estimates for fractional Schrodinger or something*, preprint.

[16] M. B. Erdoğan, and W. R. Green, *The L^p-continuity of wave operators for higher order Schrödinger operators,* Adv. Math. 404 (2022), Paper No. 108450.

[17] M. B. Erdoğan and W. R. Green, A note on endpoint L^p-continuity of wave operators for classical and higher order Schrödinger operators, J. Differential Equations **355** (2023), 144–161.

[18] M. B. Erdoğan, W. R. Green and K. LaMaster, *L^p-continuity of wave operators for higher order Schrödinger operators with threshold eigenvalues in high dimensions*, Discrete Contin. Dyn. Syst. **45** (2025), no. 9, 3258–3274.

[19] M. B. Erdoğan, W. R. Green and K. LaMaster, *L^p boundedness of wave operators for higher order Schrödinger operators with threshold eigenvalues*, preprint 2025, 29pp. arXiv:2506.16378

[20] M. B. Erdoğan, W. R. Green, and E. Toprak, *On the Fourth order Schrödinger equation in three dimensions: dispersive estimates and zero energy resonances.* J. Differ. Eq., 267, (2019), no. 3, 1899–1954.

[21] B. Feng and H. Zhang, *Stability of standing waves for the fractional Schrödinger-Hartree equation*, J. Math. Anal. Appl. **460** (2018), no. 1, 352–364

[22] H. Feng, A. Soffer, Z. Wu, and X. Yao, *Decay estimates for higher order elliptic operators*, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2805—2859.

[23] K. Fujiwara, V. S. Georgiev and T. Ozawa, *On global well-posedness for nonlinear semirelativistic equations in some scaling subcritical and critical cases*, J. Math. Pures Appl. (9) **136** (2020), 239–256

[24] A. Galtbayar and K. Yajima, *The L^p-boundedness of wave operators for fourth order Schrödinger operators on* $\mathbb{R}^4$, J. Spectr. Theory **14** (2024), no. 1, 271–354.

[25] R. Ghanmi, S. M. Boulaaras and T. Saanouni, *Local well-posedness of the fractional inhomogeneous NLS in Sobolev spaces*, Discrete Contin. Dyn. Syst. Ser. S **17** (2024), no. 10, 3035–3045

[26] M. Goldberg and W. Green, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities.* Adv. Math. 303 (2016), 360–389.

[27] M. Goldberg, and W. Green, *On the L^p boundedness of the Wave Operators for fourth order Schrödinger operators,.* Trans. Amer. Math. Soc. 374 (2021), 4075–4092.

[28] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation, J. Funct. Anal.* **133** (1995), no. 1, 50–68

[29] M. Goldberg, and M. Visan. *A Counterexample to Dispersive Estimates.* Comm. Math. Phys. 266 (2006), no. 1, 211-238.

[30] Z. H. Guo and Y. Wang, *Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations*, J. Anal. Math. **124** (2014), 1–38.

[31] W. Green, and E. Toprak, *On the Fourth order Schrödinger equation in four dimensions: dispersive estimates and zero energy resonances,* J. Differential Equations, 267, (2019), no. 3, 1899–1954.

[32] Q. Guo and S. Zhu, *Sharp threshold of blow-up and scattering for the fractional Hartree equation*, J. Differential Equations **264** (2018), no. 4, 2802–2832.

[33] X. Y. Guo and M. Y. Xu, *Some physical applications of fractional Schrödinger equation*, J. Math. Phys. **47** (2006), no. 8, 082104, 9 pp.

[34] Z. H. Guo, Y. Sire, Y. Wang, and L. Zhao. *On the energy-critical fractional Schrödinger equation in the radial case*, Dyn. Partial Differ. Equ. **15** (2018), no. 4, 265–282

[35] Y. Hong and Y. Sire, *On fractional Schrödinger equations in Sobolev spaces*, Commun. Pure Appl. Anal. **14** (2015), no. 6, 2265–2282

[36] L. Hörmander, *The existence of wave operators in scattering theory.* Math. Z. 146 (1976), no. 1, 69–91.

[37] A. Ishida, J. Lőrinczi and I. Sasaki, *Absence of embedded eigenvalues for non-local Schrödinger operators*, J. Evol. Equ. **22** (2022), no. 4, Paper No. 82, 30 pp.

[38] A. Ishida and K. Wada, *Threshold between short and long-range potentials for non-local Schrödinger operators*, Math. Phys. Anal. Geom. **23** (2020), no. 3, Paper No. 32, 11 pp.

[39] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions results in $L^2(R^m)$,* $m \geq 5$. Duke Math. J. 47 (1980), no. 1, 57–80.

[40] A. Jensen, and K. Yajima, *A remark on L^p-boundedness of wave operators for two-dimensional Schrödinger operators.* Comm. Math. Phys. 225 (2002), no. 3, 633–637.

[41] A. Jensen, and K. Yajima, *On L^p boundedness of wave operators for 4-dimensional Schrödinger operators with threshold singularities.* Proc. Lond. Math. Soc. (3) 96 (2008), no. 1, 136–162.

[42] S. Kuroda, *Scattering theory for differential operators. I.* J. Math. Soc. Japan 25 (1973), 75–104.

[43] S. Kuroda, *Scattering theory for differential operators. II. Self-adjoint elliptic operators.* J. Math. Soc. Japan 25 (1973), 222–234.

[44] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), no. 4-6, 298–305

[45] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E (3) **66** (2002), no. 5, 056108, 7 pp.

[46] Longhi, S. *Fractional Schrödinger equation in optics.* Opt. Lett. 40, 1117–1120 (2015).

[47] H. Mizutani, *Wave operators on Sobolev spaces.* Proc. Amer. Math. Soc. 148 (2020), no. 4, 1645–1652.

[48] H. Mizutani, Z. Wan, and X. Yao, *L^p-boundedness of wave operators for fourth-order Schrödinger operators on the line*,Adv. Math. **451** (2024), Paper No. 109806, 68 pp.

[49] H. Mizutani, Z. Wan and X. Yao, *Counterexamples and weak* $(1,1)$ *estimates of wave operators for fourth-order Schrödinger operators in dimension three*, J. Spectr. Theory **14** (2024), no. 4, 1409–1450.

[50] H. Mizutani, Z. Wan and X. Yao, *L^p-boundedness of wave operators for fourth order Schrödinger operators with zero resonances on* $\mathbb{R}^3$, J. Funct. Anal. **289** (2025), no. 8, Paper No. 111013, 68 pp.

[51] C. Peng and Q. H. Shi, *Stability of standing wave for the fractional nonlinear Schrödinger equation*, J. Math. Phys. **59** (2018), no. 1, 011508, 11 pp.

[52] M. Schechter, *Scattering theory for pseudodifferential operators,* Quart. J. Math. Oxford Ser. (2) 27 (1976), no. 105, 111–121.

[53] M. Schechter, *Scattering theory for elliptic operators of arbitrary order.* Comment. Math. Helv. 49 (1974), 84–113.

[54] L. Wei and Z. W. Duan, *Scattering for the fractional magnetic Schrödinger operators*, Acta Math. Sci. Ser. B (Engl. Ed.) **44** (2024), no. 6, 2391–2410

[55] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators.* J. Math. Soc. Japan 47 (1995), no. 3, 551–581.

[56] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators. II. Positive potentials in even dimensions $m \geq 4$.* Spectral and scattering theory (Sanda, 1992), 287–300, Lecture Notes in Pure and Appl. Math., 161, Dekker, New York, 1994.

[57] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators. III. Even-dimensional cases $m \geq 4$.* J. Math. Sci. Univ. Tokyo 2 (1995), no. 2, 311–346.

[58] K. Yajima, *The L^p Boundedness of wave operators for Schrödinger operators with threshold singularities I. The odd dimensional case.* J. Math. Sci. Univ. Tokyo 13 (2006), 43–94.

[59] K. Yajima, *Wave Operators for Schrödinger Operators with Threshold Singularities, Revisited.* Preprint, arXiv:1508.05738.

[60] K. Yajima, *Remark on the L^p-boundedness of wave operators for Schrödinger operators with threshold singularities*, Documenta Mathematica 21 (2016), 391–443.

[61] K. Yajima, *On wave operators for Schrödinger operators with threshold singularities in three dimensions.* Tokyo J. Math. 41 (2018), no. 2, 385–406.

[62] R. Zhang, T. Huang and Q. Zheng, *The scattering of fractional Schrödinger operators with short range potentials*, J. Funct. Anal. **281** (2021), no. 2, Paper No. 109033, 44 pp.

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