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Fractional Fourier transform applied to spatial filtering in the Fresnel domain

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Abstract

The fractional Fourier transform can be optically defined through a phase-space coordinate rotation of the Wigner distribution function associated with the input signal. This operation can be achieved by performing three successive shearing processes, which are reduced to a free-space Fresnel diffraction originated by a scaled version of the input object illuminated with a spherical wave. This result is applied to describe the behavior of spatial filtering devices based on the self-imaging phenomenon (Fresnel spatial filters).

Light propagation in a medium of position-varying refractive index can be described by analyzing the effect of an operator acting on the input field amplitude. For the case of free-space propagation (or constant refractive index), the Kirchhoff integral provides an adequate operator which can be approximated depending on the involved distances by the Fresnel and Fraunhofer integrals. Thus, the diffraction process leads to a Fourier transform relationship, operation which is the basis for developing spatial filtering devices. On the other hand, for a quadratic graded index (GRIN) medium the amplitude distribution at different locations along the propagation axis can be obtained from a fractional Fourier transform operation applied to the input signal, an approach introduced in optics by Mendlovic and Ozaktas [1–3]. When light propagates a certain finite distance inside such media, an ordinary Fourier transform is accomplished same as in the far field free-space diffraction. However, the intermediate states of the field amplitude, where the spatial and the spectral information is mixed, are different for the free-space and the GRIN media. By using an alternative definition based on considerations about the Wigner distribution function (WDF) [4–6], Lohmann suggested how to obtain the fractional Fourier transform through a proper combination of free-space propagation and lens action [7]. The equivalence between both optical definitions was later established [8].

For a certain kind of input objects that can be expressed as a linear superposition of binary grating structures, free-space diffraction in the Fresnel region was used for developing spatial filtering devices based on the self-imaging phenomenon [9–11]. In this paper, we propose a description of these Fresnel spatial filters by establishing the relationship between the Talbot conditions that are found at the self-image planes and the fractional order of the Fourier transform associated with the grating component which is cancelled as result of the interaction filter-diffracted field.

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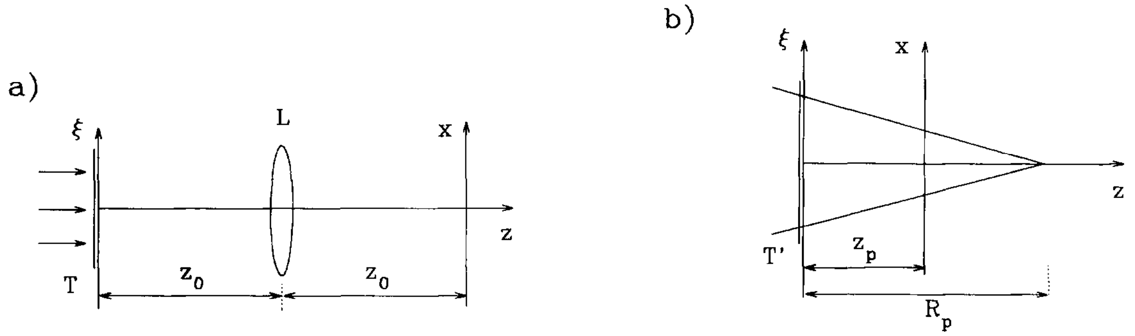

Fig. 1. (a) Optical setup for performing a fractional Fourier transform. (b) Equivalent configuration using free-space Fresnel diffraction.

From the viewpoint of the WDF, since an ordinary ($p = 1$) Fourier transform results from a $\pi/2$ coordinate rotation of the input WDF, the fractional Fourier transform of order p can be achieved through a WDF rotation in the phase-space by an angle $\phi = p\pi/2$. As was pointed out in [7], this coordinate rotation can be obtained by performing three successive shearing operations. Taking into account that WDF shearing in the spatial coordinate means free-space propagation while WDF shearing in the spatial frequency corresponds to a lens passage, very simple optical processors can be developed to optically realize the fractional Fourier transform [7,12,13]. Fig. 1a shows one of these possible optical arrangements. By combining Fresnel diffraction theory together with the link relationships to get the ϕ-rotation of the WDF, the amplitude distribution at the output plane $z = 2z_0$ becomes

$$u_p(x,y) = A\exp\left(\frac{\mathrm{i}\pi(x^2+y^2)}{\lambda f_0\tan(\phi)}\right)\iint_{-\infty}^{\infty} t(\xi,\eta)\exp\left(\frac{\mathrm{i}\pi(\xi^2+\eta^2)}{\lambda f_0\tan(\phi)}\right)\exp\left(-\frac{2\pi i(x\xi+y\eta)}{\lambda f_0\sin(\phi)}\right)\mathrm{d}\xi\,\mathrm{d}\eta, \quad (1)$$

where A is a constant, $f = f_0/\sin(\phi)$ is the focal length of L, $z_0 = f_0\tan(\phi/2)$, and $t(\xi,\eta)$ is the object amplitude transmittance. In these conditions, $u_p(x,y)$ provides the fractional Fourier transform of the input transparency; i.e.,

$$u_p(x,y) = \mathcal{F}^{(p)}\{t(\xi,\eta)\}; \qquad \phi = p\,\pi/2. \quad (2)$$

As it was mentioned above, Fresnel diffraction patterns and fractional Fourier transforms arise from light propagation in free-space and quadratic graded index media, so there is not a direct correspondence between both kind of operations. However, by performing a proper scaling of both the illuminating beam and the input transparency, that relationship between the Fresnel and the fractional Fourier transforms can be established. By simple inspection of Eq. (1), the amplitude $u_p(x,y)$ can also be rewritten as

$$u_p(x,y) = A\exp\left(\frac{\mathrm{i}\pi}{\lambda z_p}(x^2+y^2)\right)\iint_{-\infty}^{\infty} t\left(\frac{\xi}{M},\frac{\eta}{M}\right)$$
$$\times\exp\left(-\frac{\mathrm{i}\pi}{\lambda R}(\xi^2+\eta^2)\right)\exp\left(\frac{\mathrm{i}\pi}{\lambda z_p}(\xi^2+\eta^2)\right)\exp\left(-\frac{2\pi\mathrm{i}}{\lambda z_p}(x\xi+y\eta)\right)\mathrm{d}\xi\,\mathrm{d}\eta, \quad (3)$$

whenever the following relationships hold:

$$z_p = f_0\tan(\phi), \qquad M = 1/\cos(\phi), \qquad R = 2f_0/\sin(2\phi); \qquad \phi = p\,\pi/2. \quad (4)$$

Therefore, the field amplitude $u_p(x,y)$ which represents the fractional Fourier transform of the input object $t(\xi,\eta)$ for a given order p can also be thought as the Fresnel pattern, located at $z = z_p$, diffracted by a magnified object $t'(\xi,\eta) = t(\xi/M,\eta/M)$ when it is illuminated by a spherical wave converging at $z = R$ (see

Fig. 1b). The link between the value of p and the geometrical parameters involved for obtaining the Fresnel pattern is given by Eqs. (4).

Let us assume a certain object transparency $t(\xi)$ formed as a linear superposition of binary structures like Ronchi gratings. For simplicity, we follow a one-dimensional analysis. We can express $t(\xi)$ as

$$t(\xi) = \sum_m a_m R(\xi; d_m), \tag{5}$$

where

$$R(\xi; d_m) = \mathrm{sign}[\sin(2\pi\xi/d_m)]\,\mathrm{rect}(\xi/x_0), \qquad m = 1, 2, \ldots \tag{6}$$

are the Rademacher functions [14]. They are square-waves defined inside a finite interval $|\xi| < x_0$, and with a period: $d_m = 2^{2-m}x_0$. As it was studied in [9,10], these functions give rise in free-space propagation to self-imaging. Fresnel spatial filters can be developed based on this effect: a replica of the transparency $t(\xi)$ placed at a negative Talbot plane of a certain $R(\xi; d_m)$ cancels this Rademacher component so producing a filtered image of $t(\xi)$.

This filtering effect can be related to the fractional Fourier transform approach by considering Eqs. (1)-(4). Taking into account Eq. (2) and the geometry illustrated in Fig. 1b, with $t(\xi)$ (given by Eq. (5)) as the input object, we obtain at $z = z_p$:

$$u_p(x) = \mathcal{F}^{(p)}\{t(\xi)\} = \sum_m a_m \mathcal{F}^{(p)}\{R(\xi; d_m)\} = \sum_m a_m F_{z_p,R}\{R(\xi/M; d_m)\}, \tag{7}$$

where $F_{z_p,R}\{\ \}$ means a Fresnel transform, at a distance z_p, when it is employed a spherical wave converging at a distance R from the object (see Eq. (3)). The self-imaging conditions for spherical illumination are

$$\frac{1}{z_0} = \frac{1}{R} + \frac{1}{2\alpha d^2/\lambda}, \qquad \alpha = 1, 2, \ldots \tag{8}$$

where z_0 is the Talbot distance, and d is the spatial period of the object grating. By combining Eqs. (4), (7) and (8), we conclude that, at $z_p = z_0$, the fractional Fourier transform of order p for a given Rademacher function $R(\xi; d_m)$ coincides with a self-image of the scaled version of the function; i.e.,

$$\mathcal{F}^{(p)}\{R(\xi; d_m)\} = F_{z_0,R}\{R(\xi/M; d_m)\}, \tag{9}$$

where the change of scale implies: $R(\xi/M; d_m) = R(\xi; d'_m = d_m/\cos(p\pi/2))$. From the self-imaging properties for spherical illumination, we have

$$\mathcal{F}^{(p)}\{R(\xi; d_m)\} = F_{z_0,R}\{R(\xi; d'_m)\} = R(\xi; d''_m), \tag{10}$$

where

$$d''_m = d'_m(R - z_p)/R = d_m \cos(p\pi/2). \tag{11}$$

On other hand, by taking into account Eq. (10) and using Eqs. (4) and (8), we achieve

$$z_p = 2\alpha d_m^2/\lambda, \qquad \alpha = 1, 2, \ldots \quad , \tag{12}$$

which is the Talbot distance of the grating $R(\xi; d_m)$ under plane wave illumination.

In summary, for a certain object $t(\xi)$ composed of pieces with Rónchi rulings (i.e., Rademacher functions), the amplitude distribution corresponding to: $\mathcal{F}^{(p)}\{R(\xi; d_m)\}$ is identical to the amplitude associated with the

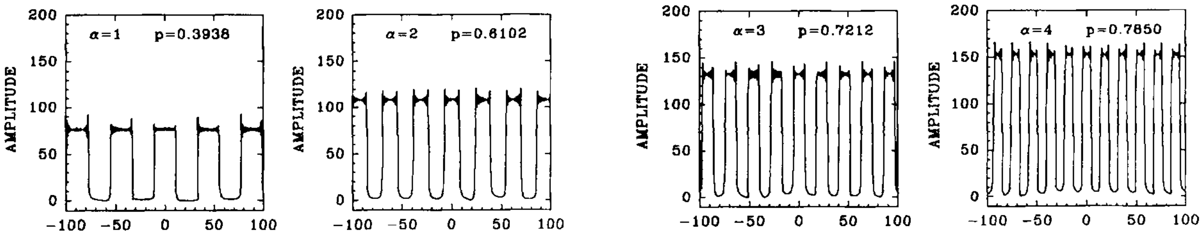

Fig. 2. Amplitude distributions associated with the fractional Fourier transform for four different values of p, which coincide with Talbot planes.

αth self-image of the grating $R(\xi; d'_m)$, illuminated with a spherical wave converging at $z = R = 2f_0/\sin(p\pi)$. The location of the Talbot/fractional Fourier transform plane: $z = z_p = f_0 \tan(p\pi/2)$ coincides with the Talbot distance: $z_0 = 2\alpha d_m^2/\lambda$, of the grating $R(\xi; d_m)$ under plane wave illumination. Thus, the relationship given by Eq. (10) links the spatial frequency $1/d''_m$ which is "blocked" by the Fresnel spatial filter action and the order p of the fractional Fourier transform associated with that grating component.

We illustrate this approach showing in Fig. 2 the amplitude distributions associated with the fractional Fourier transform of a binary grating (i.e., with a fixed spatial period d_m), for four different values of p. These values satisfy the relationship given by Eq. (12), for $\alpha = 1, 2, 3$ and 4, respectively. In this way, the resulting field amplitudes coincide with the Fresnel patterns corresponding to the first four self-images of scaled grating (as given by Eq. (10)), where the spatial periods are: $d'_m = d_m \cos(p\pi/2)$. As can be observed, the finite extension of the grating gives rise to some diffraction noise in the self-image structures.

In this paper we have established the relationship which link the fractional Fourier transform with the self-imaging phenomenon (wavefield periodicities in free-space propagation). This analysis was employed to describe the behavior of spatial filtering devices in the Fresnel domain. This approach is not so general as the Fourier approach, but it is quite simple to implement and to vary, whenever applicable. Among different applications that can be implemented using this technique we mention: analysis of the diffracted field in a periodic multiple aperture system such as the in-line analogue of the Fabry–Pérot interferometer, and microscopy of pseudoperiodic structures which can be met in biology (living cells) or metallography (alloys).

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