Can a Bicycle Create a Unicycle Track?

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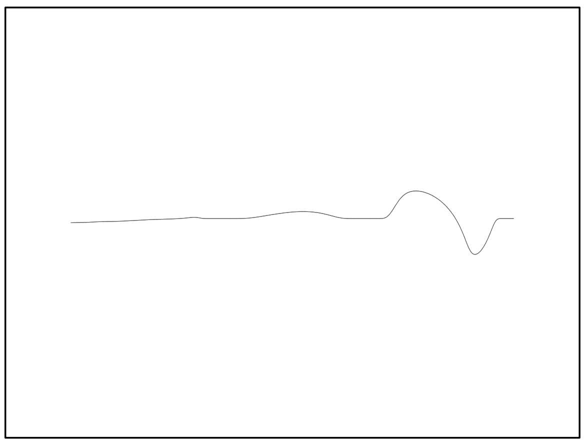

Figure 1: Can a bicycle create this single tire track?

1 Introduction

In the Sherlock Holmes mystery "*The Priory School*," a telling piece of evidence comes from the observation of a pair of tire tracks and the ensuing analysis of these tracks. This evidence leads Holmes and Watson to discover the murder of a school master and subsequently to discover the identity of the kidnapper of a Duke's son. In analyzing these tire tracks, Holmes argues that the direction the bicycle was travelling can be determined by using the indentation of the tracks at the crossings of the tire tracks to determine which is the back tire and then the direction the bicycle was travelling can be determined from the orientation of the treads of the back tire. Holmes' argument has since been refuted in the solution to the title problem of the book *"Which way did the bicycle go?"* [8], where the solution shows how the direction a bicycle is travelling can be determined using a simple calculus argument.

We consider in this paper the following variation in the Holmes mystery: *Imagine that Holmes and Watson discover a single tire track, as in Figure 1,* *instead of a pair of tire tracks, as in Figure 2, while walking down a path at the Priory School. How would Holmes and Watson determine which way the cyclist was heading?* In this situation, it seems natural to suspect that Watson would come to the obvious conclusion that the track was created by a unicycle, and thus the direction the cyclist was heading can not be determined. However, we suspect that Holmes might have drawn on his famous question to Watson, *"How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?,"* to ask our title question.

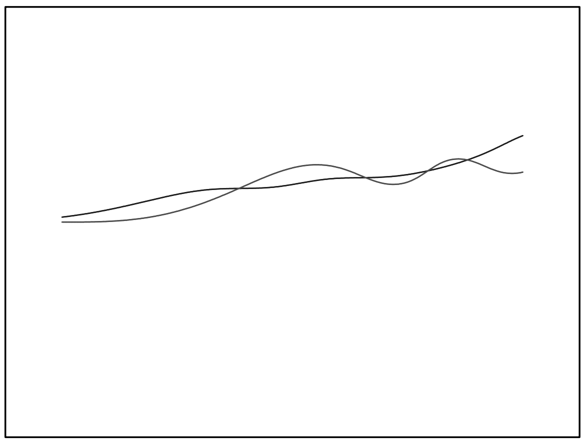

Figure 2: A pair of tire tracks.

Most of the people that we have asked the question, *"Can a bicycle create a single tire track?"*, have immediately answered *NO*, at least once we have eliminated the possibility of a single straight tire track. A few people suspect that we have asked them a trick question, and their reasoning is that a skilled rider can ride a bicycle like a unicycle by keeping the front tire in the air (a wheelie). Once this possibility has been eliminated, the answer seems to be emphatically, *NO*. A very few still suspect that this is a trick question, but they can not find the trick. These last few are of course correct, but the trick involves nothing other than mathematics.

Our goal in this paper is simple. We wish to provide a convincing argument to anyone with a solid understanding of calculus that it is possible to create a single tire track with a bicycle. We will not provide the formal existence proof here, as it involves establishing the existence of a global solution to a differential difference equation. This will be investigated in a separate paper [4]. Here, we use a more geometrical approach combined with a numerical construction. Our hope is that after reading this article the reader will be able to understand and alter the numerical algorithm outlined in Section 6 and provided in a Maple worksheet at the author's webpage [5] to produce their own examples of unicycle tracks that can be created a bicycle.

2 Physical Constructions

We will begin by describing a physical method for constructing a unicycle track with a bicycle. This construction method is based on a simple method for checking whether a given single tire track could have been created by a bicycle. In our description, we will represent the given unicycle track with a parametric curve γ parameterized with respect to its arc-length s. Furthermore, we will describe the tire tracks of a bicycle with a pair of parametric curves α and β which we will parameterize with respect to time, where $\alpha(t)$ represents the position of the point of contact of the front tire and the ground at time t and $\beta(t)$ represents the position of the point of contact of the back tire and the ground at time t.

The physical check for whether a bicycle can construct a unicycle track requires that we be able to place both tires of the bicycle on the unicycle track at the same time, and be able to push the bicycle on the unicycle track keeping both bicycle tires in contact with the unicycle track at all times. This means that for each time t there must exist positions $\gamma(s_1(t))$ and $\gamma(s_2(t))$ on the unicycle track such that $\beta(t) = \gamma(s_1(t))$ and $\alpha(t) = \gamma(s_2(t))$. We can now check whether a bicycle can create a unicycle track by first checking whether we can place the bicycle on the unicycle track at time $t = 0$, i.e. do there exist points $\gamma(s_1(0))$ and $\gamma(s_2(0))$ such that $\beta(0) = \gamma(s_1(0))$ and $\alpha(0) = \gamma(s_2(0))$? Next, we push the bicycle on the unicycle track in such a manner that the front tire remains in contact with the unicycle track, i.e. $\alpha(t) = \gamma(s_2(t))$ for some function $s_2(t)$. If it happens that the back tire remains in contact with the unicycle track, i.e. $\beta(t) = \gamma(s_1(t))$ for some function $s_1(t)$, then the unicycle track could have been created by a bicycle. We note that we could have just as easily pushed the bicycle on the unicycle track in such a manner that the back tire remains in contact with the unicycle track, and tested whether the front tire track remains in contact with the unicycle track.

Our physical construction method is based on the following variation of the physical check. Start with a segment of a unicycle track $\gamma(s)$ with $s_1 < s < s_2$, and suppose we can place the bicycle on this track segment with $\alpha(0) = \gamma(s_2)$ and $\beta(0) = \gamma(s_1)$. Now, push the bicycle forward on the curve segment keeping the back tire of the bicycle in contact with the unicycle track, and extend the segment of the unicycle track by constructing $\alpha(t)$ from $\beta(t)$. Next, push the bicycle backward on the curve segment keeping the front tire of the bicycle in contact with the unicycle track, and extend the segment of the unicycle track by constructing $\beta(t)$ from $\alpha(t)$. This process of pushing the bicycle forward or backward on the unicycle track may be continued indefinitely, provided the front or back tire can remain in contact with the curve segment created, i.e. if the curve segment created is differentiable with respect to arc-length. Therefore, given a suitable initial segment, we can create a unicycle track with a bicycle. This is of course provided we can determine the position of the front tire given the back tire track, and the position of the back tire given the front tire track.

The main goal of the remainder of this paper is to describe the physical construction in mathematical terms. This requires an analysis of the geometry of bicycle tracks, which provides us with the equations necessary to mathematically construct a bicycle track and the relations necessary to construct an initial curve segment to start our construction process. We want to emphasize that constructing the initial curve segment is the technically hard part of our construction, as we need the initial segment to satisfy an infinite number of conditions. In fact, most curve segments will not generate a unicycle track with our methods. For instance, applying our construction to the curve segment $c(t) = [t, t^2(1-t)^2]$ with $0 \le t \le 1$ produces the curve in Figure 3, where the darker thicker segment is the extension produced by pushing the bicycle forward. Notice that the curve in Figure 3 is not smooth, specifically at the point where the forward direction meets the initial segment, which means that we can not continue to push the bicycle forward any farther without destroying the continuity of the track.

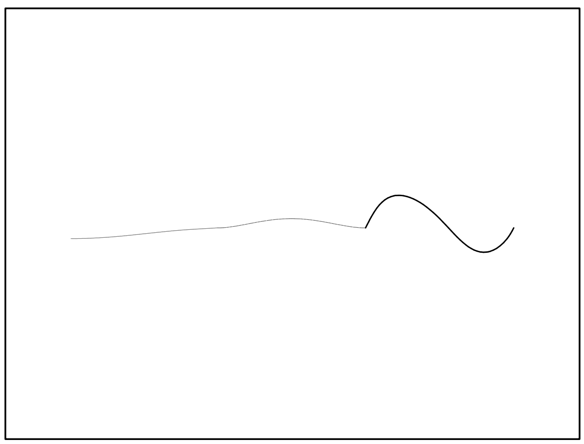

Figure 3: Curve generated with initial track $c = [t, t^2(1-t)^2]$.

3 Geometry of Bicycle Tracks

The construction process requires that we be able to generate the front tire track from the back tire track, and generate the back tire track from the front tire track. This entails deriving equations which relate the positions and unit tangent vectors of the front and back tire tracks, and the curvature of both the front and back tire tracks. To state these equations, we will need to use some elementary differential geometry, and some basic facts about bicycles in order to derive these equations. Most of the differential geometry that we need can be found in calculus textbooks, for instance [3] or [10]. For more complete details on the subject of differential geometry, we refer the reader to the texts [2], [7] and [9].

We will assume for the rest of this paper that the bicycle is ridden on a perfectly flat surface, so that the curves α, β, and γ will be plane curves. We will also assume the plane of each bicycle tire meets the ground (the surface on which the bicycle is ridden) in a right angle. This assumption simplifies the analysis by removing the angle at which the bicycle is banked into the ground, the angle between the plane of the bicycle frame and the plane of the ground. We suspect that it is possible to construct a unicycle track with a bicycle without these assumptions, but the analysis would be much harder as the equations governing our construction would then depend on the geometry of the surface and the angle at which the bicycle is banked.

Using the above assumptions on the bicycle and the ground, the position of the front tire $\alpha(t)$ can be easily expressed in term of the back tire's position $\beta(t)$ and direction $T_\beta(t)$ as

$$\alpha(t) = \beta(t) + l\, T_\beta(t), \tag{3.1}$$

where l is a positive constant representing the length of the bicycle and $T_\beta(t)$ is the unit tangent vector of the back tire track β at time t. This equation is a direct result of two simple observations concerning the manner in which a bicycle is built. The first observation is that the back tire is fixed in the frame, meaning that the back tire and the frame are aligned. This means that the tangent line to the back track at time t should be equal to the secant line through the front and back tire positions at time t, i.e.

$$T_\beta(t) = \frac{\beta'(t)}{|\beta'(t)|} = \frac{\alpha(t) - \beta(t)}{|\alpha(t) - \beta(t)|}. \tag{3.2}$$

The second observation is that the frame is rigid, meaning that the distance $|\alpha(t) - \beta(t)|$ between the points of contact between the tires and the ground is a constant independent of the positions of the tires (provided that the bicycle is ridden on a flat surface), so $|\alpha(t) - \beta(t)| = l$. To simplify the remaining equations, we will now assume that $l = 1$, which means in geometrical terms that our unit of measurement is a bike length.

Remark. *For convenience, we will denote the intrinsic geometric properties of the curves α, β and γ by using subscripts. The unit tangent of α will be denoted by T_α, the unit tangent of β will be denoted by T_β, and the unit tangent of γ will be denoted by T_γ. When there is no subscript, we are referring to quantities of a generic plane curve.*

Remark. *We will usually suppress the dependence of all quantities on the parameters, with the exception being when we differentiate. Then, we will use differential notation, dT/ds or dT/dt.*

It is worth noting that while riding a bicycle one does not have direct control of the back tire's direction (the direction of the frame) or position. One only has direct control over the direction of the front tire (at least with respect to the frame). For this reason, it is useful to have an expression for the back tire's position in terms of the front tire's position and direction in order to analyze the geometry of bicycle tracks. We accomplish this by introducing an angle Θ, which represents the angle between the front tire track's unit tangent T_α and the frame (the back tire track unit tangent) T_β. We require that this angle Θ be signed and satisfy $-\frac{\pi}{2} \leq \Theta \leq \frac{\pi}{2}$, with a positive angle representing a left turn and a negative angle representing a right turn.

To quantify left and right in terms of mathematics, we use the principal unit normal vector of a plane curve. The principal unit normal of a plane curve c is defined to be the unit vector N orthogonal to the unit tangent vector T such that the ordered pair of vectors $[T, N]$ is a rotation of the standard basis vectors $[\mathbf{i}, \mathbf{j}]$ of the plane. This means that if T is given in terms of the standard basis vectors $[\mathbf{i}, \mathbf{j}]$ as

$$T = \cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j},$$

then N is given in terms of $\mathbf{i}$ and $\mathbf{j}$ by rotating T by an angle of $\frac{\pi}{2}$ radians, i.e.

$$\begin{aligned} N &= \cos(\phi + \frac{\pi}{2})\,\mathbf{i} + \sin(\phi + \frac{\pi}{2})\,\mathbf{j} \\ &= -\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j}. \end{aligned} \tag{3.3}$$

Notice that N is given by the standard trick for obtaining a vector orthogonal to $T = a\,\mathbf{i} + b\,\mathbf{j}$ by $N = -b\,\mathbf{i} + a\,\mathbf{j}$.

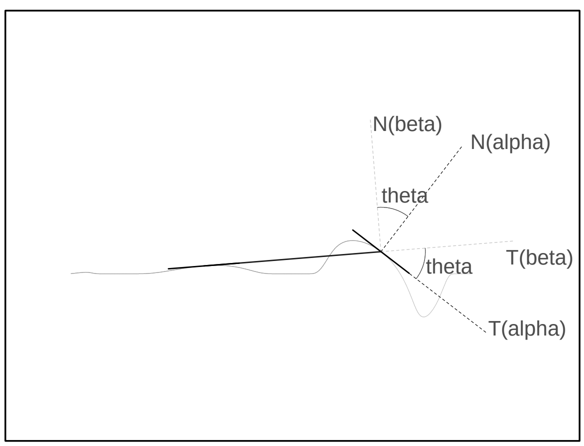

Figure 4: Relations between T_α, N_α, T_β, N_β.

One of the principal uses of the angle Θ is to convert between the two sets of vectors associated with the bicycle, $[T_\alpha, N_\alpha]$ and $[T_\beta, N_\beta]$. This conversion will be used to compute the position of the back tire given the front tire. Using basic trigonometry and vector arithmetic, we have that

$$\begin{aligned} T_\alpha &= \cos\Theta\, T_\beta + \sin\Theta\, N_\beta \\ N_\alpha &= -\sin\Theta\, T_\beta + \cos\Theta\, N_\beta \end{aligned} \quad \text{and} \quad \begin{aligned} T_\beta &= \cos\Theta\, T_\alpha - \sin\Theta\, N_\alpha \\ N_\beta &= \sin\Theta\, T_\alpha + \cos\Theta\, N_\alpha \end{aligned} \tag{3.4}$$

Thus, by combining (3.1) and (3.4) we can write

$$\beta = \alpha - (\cos\Theta\, T_\alpha - \sin\Theta\, N_\alpha)\,, \tag{3.5}$$

which gives the back tire's equation in terms of the front tire's position and direction.

The equations (3.1) and (3.5) only allow us to determine the equation of one of the tire tracks given the other tire track. This is useful for determining which direction the bicycle was travelling, as these equations provide a simple physical check (see [8] page ?), much like the physical check described in Section 2. However, these equations do not permit us to mathematically create a bicycle track. In order to do this, we need to derive a system of differential equations for creating the tire tracks of a bicycle. These differential equations are easiest to derive using a geometric approach. The key fact upon which the derivation relies is the remarkable feature of plane curves that a plane curve is uniquely described up to a rotation and translation by its *signed curvature*. For completeness, we include a brief description of this remarkable fact, known as the fundamental theorem of plane curves. We refer the interested reader to the texts [2], [7] and [9] for more details.

The fundamental theorem of plane curves is a simple consequence of how the unit tangent vector T of the curve and the principal unit normal vector N change with respect to the parameter of the curve. In fact, it is a simple exercise in vector calculus ([10] page 725 Exercises 45, 46) to derive the *Frenet frame equations*,

$$\frac{dT}{ds} = \kappa\, N \quad \text{and} \quad \frac{dN}{ds} = -\kappa\, T, \tag{3.6}$$

from the fact that T and N are orthogonal unit vectors, where κ is the signed curvature of the curve and s is the arc-length of the curve. A curve c with signed curvature κ is given by solving the Frenet frame equations (3.6) as follows. First, write the unit tangent vector T as $T = \cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j}$, so the signed curvature is given by $\kappa = d\phi/ds$. Then, a curve c with signed curvature κ is given by

$$c = \int (\cos(\phi)\,\mathbf{i} + \sin(\phi))\,\mathbf{j})\, ds \quad \text{where} \quad \phi = \int \kappa\, ds. \tag{3.7}$$

In (3.7), we have suppressed information in the integral signs for convenience. We should have used definite integrals and initial conditions rather than indefinite integrals.

With the fundamental theorem of plane curves, we can now derive the differential equations for creating bicycle tracks. These equations come from differentiating (3.1) and (3.4). By differentiating (3.1) with respect to t using the chain rule and the Frenet frame equations, we get

$$\frac{d\alpha}{dt} = T_\alpha \frac{ds_\alpha}{dt} = T_\beta\, \frac{ds_\beta}{dt} + \kappa_\beta\, N_\beta\, \frac{ds_\beta}{dt}, \tag{3.8}$$

where κ_β is the curvature of β, and s_α and s_β are the arc-length parameters of the curves α and β. Equation (3.8) implies

$$\frac{ds_\alpha}{dt} = \sqrt{1 + (\kappa_\beta)^2}\,\frac{ds_\beta}{dt} \tag{3.9}$$

and therefore

$$T_\alpha = \frac{1}{\sqrt{1 + (\kappa_\beta)^2}}\,T_\beta + \frac{\kappa_\beta}{\sqrt{1 + (\kappa_\beta)^2}}\,N_\beta. \tag{3.10}$$

By equating the expressions for T_α in (3.4) and (3.10), we find

$$\cos\Theta = \frac{1}{\sqrt{1 + (\kappa_\beta)^2}} \quad \text{and} \quad \sin\Theta = \frac{l\kappa_\beta}{\sqrt{1 + (\kappa_\beta)^2}}, \tag{3.11}$$

which implies

$$\kappa_\beta = \tan\Theta \quad \text{and} \quad \frac{ds_\alpha}{dt} = \sec\Theta\,\frac{ds_\beta}{dt}. \tag{3.12}$$

The importance of (3.12) is that it allows us to determine the back tire track knowing only the angle Θ and the speed of the back tire, since these allow us to solve the Frenet frame equations for the back tire track. Furthermore, knowing the back tire track, we can determine the front tire track using (3.1).

The above equations are mainly used in pushing a bicycle forward in our method for creating a unicycle track with a bicycle. In our methods, we also need to be able to push the bicycle backwards, which means we need to be able to solve the Frenet frame equations to generate the front tire track and then use (3.5) to generate the back tire track. We thus need an expression for κ_α in terms of Θ. By differentiating the equation for T_α in (3.4) with respect to t, we find that

$$\kappa_\alpha\,N_\alpha\,\frac{ds_\alpha}{dt} = (-\sin\Theta\,T_\beta + \cos\Theta\,N_\beta)\left(\frac{d\Theta}{dt} + \kappa_\beta\,\frac{ds_\beta}{dt}\right), \tag{3.13}$$

which implies that

$$\kappa_\alpha\,\frac{ds_\alpha}{dt} = \frac{d\Theta}{dt} + \tan\Theta\,\frac{ds_\beta}{dt} \tag{3.14}$$

or

$$\kappa_\alpha = \cos\Theta\,\frac{d\Theta}{ds_\beta} + \sin\Theta \quad \text{or} \quad \kappa_\alpha = \frac{d\Theta}{ds_\alpha} + \sin\Theta. \tag{3.15}$$

In (3.15), we have used (3.12) to write κ_β in terms of Θ and to replace differentiation with respect to t with either differentiation with respect to s_α or s_β.

We want to emphasize the importance of equations (3.12), (3.14) and (3.15) derived in this section. These are the equations that allow us to create a bicycle track knowing only the speed of the back tire and how the front tire is turned with respect to the bicycle frame. Notice that this matches the everyday experience of riding a bicycle. In fact, we will show in [6] how to use these equations to construct bicycle tracks for which it is impossible, using the method described in [8], to determine which way the bicycle is going.

4 A Differential Difference Equation.

The equation that governs the construction of a unicycle track by a bicycle is now easy to derive. The essential facts that determine this equation are that curvature does not depend on the parameterization of the curve and that a unicycle track which can be created by a bicycle has two natural parameterizations, one given by the front tire and one given by the back tire. To state the equation, let us assume for the moment that we have a unicycle track which can be created with a bicycle. Then for each point $\gamma(s)$ on the unicycle track, there are times t_1 and t_2 with $t_2 > t_1$ such that $\alpha(t_1) = \beta(t_2) = \gamma(s)$. Therefore $\kappa_\alpha(t_1) = \kappa_\beta(t_2) = \kappa_\gamma(s)$ which implies that

$$\left(\frac{d\Theta}{ds_\alpha} + \sin\Theta\right)\Big|_{t=t_1} = \tan\Theta\Big|_{t=t_2}, \tag{4.1}$$

by setting (3.15) equal to (3.12). Notice that equation (4.1) is not a traditional differential equation for Θ, because the expression for the derivative of Θ depends on the value of Θ at two different times. Such differential equations are called differential difference equations or more generally functional differential equations.

In this paper, we will treat (4.1) like an ordinary differential equation, and solve it by using integration. However, there are some important differences. First, we need an initial segment, $\Theta(t)$ for $t_1 \leq t \leq t_2$ rather than an initial value, $\Theta(t_0)$, to determine a solution. Second, we can only solve (4.1) by integrating for $t < t_1$. Moreover, we are only guaranteed that the solution will be differentiable solution for $t < t_1$. This is due to the fact that there is no requirement that the initial segment has to be differentiable, which means that there might not be an extension of the initial segment to $t > t_2$ satisfying (4.1). This is exemplified in Figure 3, where it is easy to see the curve generated is not differentiable for $t > 1$, but it is differentiable for $t < 0$. This means that in order to generate a smooth solution for all t we will need extra compatibility conditions on the initial segment.

5 Construction of an Initial Track Segment

The equations in Sections 3 and 4 are primarily used to generate the tire tracks, once we have an appropriate initial curve segment. In this section, we show how the analysis of Section 3 also provides us with conditions on the initial segment, and that there are initial segments for which we can start our construction method. We again want to emphasize that this is the hard part in our construction, as we need the initial curve segment to satisfy an infinite number of compatibility conditions.

The conditions on the initial segment are easiest to state using ideas in our physical method, and the equations in Section 3. Specifically, if we suppose that γ is a unicycle track that can be created with a bicycle, then there are continuous functions $s_1(t)$ and $s_2(t)$ such that $\beta(t) = \gamma(s_1(t))$ and $\alpha(t) = \gamma(s_2(t))$. This allows us to rewrite (3.1) and (3.4) respectively in terms of γ as

$$\gamma(s_2) = \gamma(s_1) + T_\gamma(s_1), \tag{5.1}$$

and

$$T_\gamma(s_2) = \frac{1}{\sqrt{1+(\kappa_\gamma(s_1))^2}}\, T_\gamma(s_1) + \frac{\kappa_\gamma(s_1)}{\sqrt{1+(\kappa_\gamma(s_1))^2}}\, N_\gamma(s_1). \tag{5.2}$$

Furthermore, by differentiating (3.1) with respect to time and using (3.9) to convert from derivatives with respect to time to derivatives with respect to arc-length, we get

$$\kappa_\gamma(s_2) = \frac{1}{(\sqrt{1+(\kappa_\gamma(s_1))^2})^3}\frac{d\kappa_\gamma}{ds}\Big|_{s=s_1} + \frac{\kappa_\gamma(s_1)}{\sqrt{1+(\kappa_\gamma(s_1))^2}}. \tag{5.3}$$

Notice that (5.3) defines the signed curvature κ of γ when $s = s_2$ in terms of the signed curvature κ and the derivative of the signed curvature with respect to arc-length when $s = s_1$. Therefore, we find that the derivatives of the curvature κ with respect to arc-length when $s = s_2$ are given in terms of the curvature and the derivatives of the curvature with respect to arc-length when $s = s_1$, that is

$$\frac{d^n\kappa}{ds^n}(s_2) = F_n\left(k(s_1), \frac{d\kappa}{ds}(s_1), ..., \frac{d\kappa^{(n+1)}}{ds^{n+1}}(s_1)\right), \tag{5.4}$$

for specific functions F_n that can be determined by repeatedly differentiating (5.3). This establishes an infinite number of compatibility conditions on an initial curve segment that can be used in our construction process.

The question then is: *Does there exist a curve segment* $\gamma(s)$ *that satisfies* (5.1)*,* (5.2)*,* (5.3) *and* (5.4)*?*

Luckily, it is easy to verify that a straight line ($\kappa_\gamma \equiv 0$) satisfies these compatibility conditions. The important observation, based on the local nature of our compatibility conditions, is that we only need a curve whose curvature and all the derivatives of the curvature are zero at the two endpoint of the initial segment. This requirement is equivalent to finding a non-constant function all of whose derivatives are equal to zero at two points. Such functions exist, but they are outside the realm of analytic functions one normally encounters in calculus. It is a standard exercise in an advanced calculus or elementary real analysis course to show that all the derivatives of the function

$$f(t) = \begin{cases} 0 & \text{if } t = 0; \\ \mathrm{e}^{-1/x^2} & \text{if } t \neq 0 \end{cases} \tag{5.5}$$

when $t = 0$ are zero, but the function is not equal to a constant (see [1] for more details).

Using function like (5.5), we can define a class of infinitely differentiable functions on $\mathbb{R}$ that are identically equal to zero when $t < 0$ and when $t > 1$

but are not equal to zero in the interval $0 < t < 1$. Choose a smooth function $h(t)$ on $\mathbb{R}$ with $h(0) = h(1) = 0$ and $h(t) > 0$ for $0 < t < 1$, for instance take $h(t) = \sin^2(\pi t)$, and consider the piecewise defined function,

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq 0; \\ \mathrm{e}^{-1/h(t)} & \text{if } 0 < t < 1; \\ 0 & \text{if } t \geq 1. \end{cases} \tag{5.6}$$

The same type of calculation that shows (5.5) is infinitely differentiable, also shows that (5.6) is infinitely differentiable and all the derivatives when $t = 0$ and when $t = 1$ are zero.

It is now easy to define an initial curve for generating a unicycle track with a bicycle. Start with a parametric equation of a line $\mathbf{L}(t)$, and choose a smooth function φ so that $\varphi(t) = 0$ for $t \leq 0$ and $t \geq 1$, like (5.6). An initial curve segment can then defined by

$$\gamma(t) = \mathbf{L}(t) + \varphi(t)\mathbf{c}(t) \quad \text{for } 0 \leq t \leq 1 \tag{5.7}$$

for any smooth parametric curve $\mathbf{c}$. In fact, the initial curve to generate the unicycle track in Figure 1 was constructed using (5.7) with $h(t) = \sin^2(\pi t)$ in (5.6), and $\mathbf{L}(t) = [t, 0]$, $\mathbf{c}(t) = [t, \frac{1}{4}\sin(t)]$.

6 A Numerical Construction

In principal, we can now construct a single tire track with a bicycle from an initial segment of form (5.7) as per the physical construction in Section 2. Use (5.1) to inductively define the forward direction, and solve the differential difference equation (4.1) to define the backward direction. However, because of the inductive nature of the forward direction and the general difficulties in solving the nonlinear differential difference equation (4.1), we will use numerical methods in this section to approximate a unicycle track that can be created with a bicycle

First, we construct an initial segment as in Section 5 parameterized with respect to t on the interval $0 \leq t \leq 1$ with the condition that $d\gamma/dt \neq 0$ so the unit tangent vector T_γ is well-defined. We then sample the curve at $M + 1$ points, for instance set $t_i = i/M$ with $i = 0, 1, 2, \cdots, M$, and define $c_i = \gamma(t_i)$. Symbolically (or numerically), we then calculate the unit tangent vector T_i and the curvature κ_i for each sampled point c_i. Thus, we have $M + 1$ data sets (c_i, T_i, κ_i) from which we can obtain an approximation of the initial curve segment by interpolation. In our calculation, we will also use the principal unit normal vector N_i and the angle Θ_i, which can be computed from T_i and κ_i using $\Theta_i = \arctan(\kappa_i)$ and $N_i = -b\,\mathbf{i} + a\,\mathbf{j}$ if $T_i = a\,\mathbf{i} + b\,\mathbf{j}$.

To produce forward direction of the unicycle track is easy. We set $\beta(t_i) = c_i$, and use the equations (3.1), (3.4) and (3.15) to define (c_i, T_i, κ_i) for $i > M$ by setting $c_{i+M} = \alpha(t_i)$. To calculate κ_{i+M}, we use finite difference approximation to estimate the derivative in (3.15). Thus, we have the iterative formulas

$$\begin{cases} c_{i+M} = c_i + T_i \\ T_{i+M} = \cos(\Theta_i)\, T_i + \sin(\Theta_i)\, N_i \\ N_{i+M} = -\sin(\Theta_i)\, T_i + \cos(\Theta_i)\, N_i \\ \kappa_{i+M} = \dfrac{\Theta_{i+1} - \Theta_i}{\|c_{i+1} - c_i\|} + \sin(\Theta_i) \\ \Theta_{i+M} = \arctan(\kappa_{i+M}) \end{cases} \tag{6.1}$$

which produces the extension of the unicycle track by pushing the bicycle forward.

The backward direction requires solving (4.1) using numerical methods. We use Euler's method for simplicity, though we could use any numerical integration technique. By finite difference approximation in (3.15), we have

$$\frac{d\Theta}{ds_\alpha}\Big|_{t=t_i} \approx \frac{\Theta(t_i) - \Theta(t_{i-1})}{\|\alpha(t_i) - \alpha(t_{i-1})\|}. \tag{6.2}$$

Thus, we can rewrite (4.1) using (6.2) as

$$\Theta_{i-1} = \Theta_i + (\sin(\Theta_i) - \tan(\Theta_{i+M}))\, \|c_{i+M} - c_{i+M-1}\|. \tag{6.3}$$

To determine c_{i-1}, we use (3.5) to calculate the back tire position at time t from the front tire position and direction at time t. However, since we also need T_{i-1} to continue the process, we first use the relation in (3.4) to calculate T_{i-1} from $T_\alpha(t_{i-1}) = T_{i+M-1}$ and $N_\alpha(t_{i-1}) = N_{i+M-1}$. Therefore, we have the iterative formulas

$$\begin{cases} \Theta_{i-1} = \Theta_i + (\sin(\Theta_i) - \tan(\Theta_{i+M}))\, \|c_{i+M} - c_{i+M-1}\| \\ \kappa_{i-1} = \tan(\Theta_{i-1}) \\ T_{i-1} = \cos(\Theta_{i-1})\, T_{i+M-1} - \sin(\Theta_{i-1})\, N_{i+M-1} \\ N_{i-1} = \sin(\Theta_{i-1})\, T_{i+M-1} + \cos(\Theta_{i-1})\, N_{i+M-1} \\ c_{i-1} = c_{i+M-1} - T_{i-1} \end{cases} \tag{6.4}$$

which produces the extension of the unicycle track by pushing the bicycle backward.

We thus have a numerical method for producing data sets $\{(c_i, T_i, \kappa_i)\}$ from which we can approximate the curve by interpolation. For instance, we have used linear interpolation to create Figure 5 as an extension of the initial segment

$$\gamma(t) = [t, \varphi(t)\, t^2(1-t)^2] \tag{6.5}$$

where $\varphi(t)$ is of the form (5.6) with $h(t) = \sin^2(\pi t)$. Other interpolation methods could also be used, for instance cubic spline interpolation.

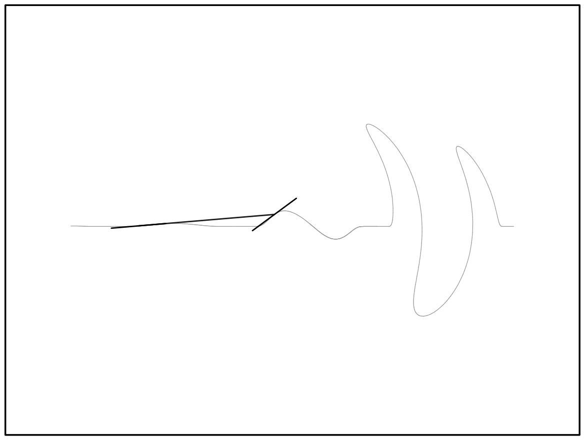

Figure 5: A single tire track that can be created by a bicycle

The potential problem with our numerical algorithm (or any numerical algorithm) is whether we have convergence as the initial sampling of the segment is refined. For the forward direction, we are guaranteed convergence, because we have a formal extension of the curve by equation (5.1). In the backward direction, we can use standard arguments in differential equations to guarantee local convergence, but global convergence is difficult. We will defer that issue to another paper [4], however our numerical method produces curves (through interpolation) that appear to converge at least for some initial curve segments. The main problem in the backward direction is guaranteeing when we solve the differential difference equation that the angle Θ satisfies $-\frac{\pi}{2} < \Theta < \frac{\pi}{2}$, so the curvature is continuous and nonsingular.

Another issue that arises once one has looked at some of the curves generated by this algorithm is whether it is possible to physically construct a unicycle track by our methods. We have so far ignored the turning radius of the bike, and assumed that the range of the angle Θ is $-\frac{\pi}{2} < \Theta < \frac{\pi}{2}$. In actuality, the turning angle is restricted by $-\epsilon < \Theta < \epsilon$ for some positive number ϵ considerably less than $\frac{\pi}{2}$. For all the unicycle tracks generated by our methods, it seems that Θ gets arbitrarily close to $\pm\frac{\pi}{2}$. This is important, because according to our construction, it should be possible to drive a car so that the front tire tracks and the back tire tracks overlap, since this is the same as two bicycles riding at the same speed and parallel to each other.

Whether or not it is physically possible, we have accomplished one of our goals. We have shown that it is mathematically possible to construct a unicycle track with a bicycle. In order to see the construction of such a track, we refer the reader to the web-page [5]. At that site, there are several animations of bicycles constructing unicycle tracks, and there is a Maple worksheet which can be downloaded for constructing unicycle tracks with a bicycle.

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