# MA 323 Geometric Modelling
# Course Notes: Day 05
# Model Analysis
# An Approximation Problem

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Over the last few days, we have looked at various methods for creating models and some of the common types of problems that modelling methods might need to solve. Today, we want to start discussing determining at least geometrically and aesthetically when we have a good model. This is the model analysis problem. For today, we will measure goodness of the model in terms of minimizing the distance between the actual object we want to model and the model we created. This means we need to define a method of measuring the distance between two curves, so we can talk about which curve is closer to the curve we are trying to model.

## 5.1 Model Analysis<br>An Approximation Problem

The model analysis problem is the last step in the modelling process. Once the model has been created, one must determine whether the model is acceptable. This means one must determine whether it is a good model or a bad model. This could be a subjective assessment, especially if one is creating a design of something that has never existed except in one's own mind. However, it could also be very objective, especially if one is trying to create a model of an existing physical or virtual object. In this case, one typically has a standard of goodness possibly based on visual sensing. One might be able to tell by looking at the model whether it is a good model or a bad model.

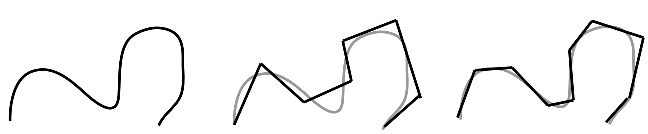

Figure 1: Approximations

At this point, we will restrict ourselves to measuring the goodness of the models with respect to distance. A model will be a good model if the model is close in distance to the object it is to represent. To explain this concept of close in distance, consider the case where we wish to model the curve $c$ in the figure above and on the left. We choose an acceptable tolerance

$\epsilon > 0$. We consider the model to be good if the distance between the model and the curve is less than $\epsilon$, see diagrams above. From the diagrams above, it is easy to tell that the model on the far right is the better of the two provided. The question, we want to answer is it within the acceptable tolerance.

Determining precisely whether a curve is within an acceptable tolerance is intuitively very simple but hard to calculate. However, we need some terminology from *analysis* and *topology* to state these concepts in a rigorous mathematical form. A *neighborhood* about the curve $c$ is formed by considering the set $\mathcal{N}_\epsilon(c)$ formed from the union of the interiors of circles of radius $\epsilon$ with centers on $c$, see diagram below. In mathematical terms, the set $\mathcal{N}_\epsilon(c)$ is given as

$$\mathcal{N}_\epsilon(c) = \cup_{t\in[0,1]}\{(x,y) : (x-\zeta(t))^2 + (y-\eta(t))^2 < \epsilon\}$$

where $c$ is parameterized over the interval $[0,1]$ and given by $x = \zeta(t)$, $y = \eta(t)$, see the diagram below on the right. For a model to be good, it is necessary that the model lie entirely within a neighborhood of acceptable tolerance, the neighborhood $\mathcal{N}_\epsilon(c)$. Moreover, the curve modelled must lie within a neighborhood of acceptable tolerance of the model.

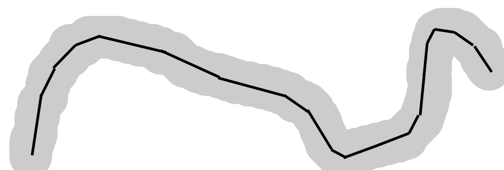

Figure 2: A Neighborhood of a Curve

The concept of the distance between two parametric curves is closely related to the concept of neighborhoods. We define the distance between two parametric curves $\alpha(t)$ and $\beta(s)$ using a maximin argument. Recall, the distance between a point and a curve is determine by finding the minimum distance between the point and a point on the curve. Thus, we can define a function $d_{\alpha,\beta}(t)$ to be the distance between the point $\alpha(t)$ and the curve $\beta$. The curve $\alpha$ lies within an $\epsilon$ neighborhood of the curve $\beta$ if $\epsilon > d_{\alpha,\beta}(t)$ for all $t$. Therefore, we say the distance from $\alpha$ to $\beta$ is then the maximum of $d_{\alpha,\beta}(t)$. Similarly, we can define $d_{\beta,\alpha}(s)$ to be the distance between the point $\beta(s)$ and the curve $\alpha$, then the curve $\beta$ lies within an $\epsilon$ neighborhood of the curve $\alpha$ if $\epsilon > \max_s d_{\beta,\alpha}(s)$. The distance from $\beta$ to $\alpha$ is then the maximum of $d_{\beta,\alpha}(s)$.

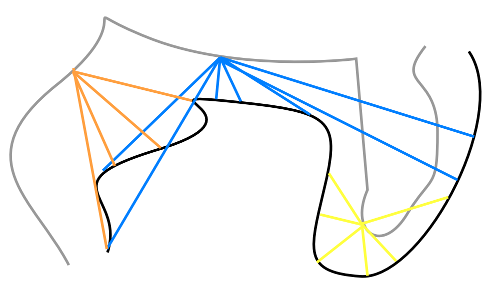

Figure 3: Distance from One Curve to Another Curve

It is worth noting that the distance from $\alpha$ to $\beta$ is not necessarily equal to the distance from $\beta$ to $\alpha$. This is a general property of the max-min arguments,

$$\max_t(\min_s \|\alpha(t) - \beta(s)\|) \neq \max_t(\min_s \|\alpha(s) - \beta(t)\|)$$

One can define the distance between the two curves in a symmetric manner, by considering

$$dist(\alpha, \beta) = \max(d_{\alpha,\beta}, d_{\beta,\alpha})$$

With this concept of distance, we have for any $\epsilon > dist(\alpha, \beta)$ we have $\alpha$ within $N_\epsilon(\beta)$ and $\beta$ within $N_\epsilon(\alpha)$.

This is just one manner of measuring closeness. In the geometric modelling, this is a coarse measure of closeness. For instance, in approximating a circle (see figure below), one can have a piecewise linear approximation given by interpolation from data points closer than a "drawn" circle. However, one can easily argue that the "drawn" circle is a better model of a circle because it appears smoother than the piecewise linear interpolant, see figure below.

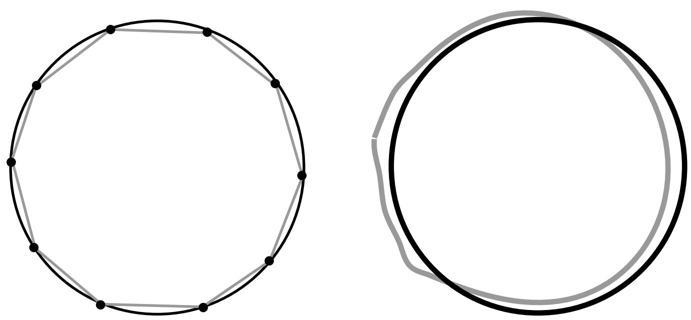

Figure 4: Comparing Two Models of a Circle

In the figure above, one can argue that a problem with the "drawn" circle could be the placement of the circle when calculating the distance. If the circle is moved slightly to the left in the figure on the right, then the "drawn circle" could be better than the piecewise linear interpolant. The measure of closeness presented above depends on the relative positions of the two objects. Technically, in comparing the models one has to minimized with respect not only to points but also the orientations of the objects, as these will affect the distance. In fact, this measure of closeness assumes that either physically of virtually the objects can be placed on top of each other to calculate the distance.

## 5.2 Better Measures of Closeness

To get a better notion of closeness requires more sophisticated mathematical description of the model, which we will discuss in more detail when we discuss more sophisticated modelling techniques. For the moment, we want to discuss briefly what better notions of closeness would entail.

The measure of closeness presented above is closeness in distance. As mentioned above, it is possible to have a curve close in distance but not a good model, see diagram below. The breakdown of this measure of closeness as given by the diagram below is in terms of wildly oscillating approximation that stays close to the circle. In terms of distance, one would have to say this is a good approximation. However, it does not portray the smoothness of the circle. The tangent lines of the approximating curve do not accurately reflect the tangents of the point on the circle. A better approximation would have the tangents agreeing also.

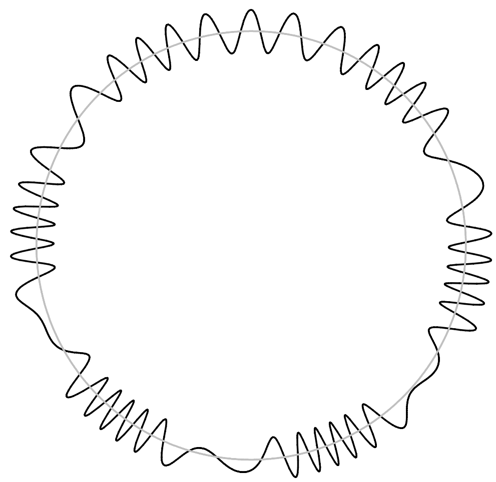

Figure 5: Example of a Model Close in Distance, but Not a Necessarily Good Model

The approximation in closeness in distance is in a sense like saying two functions have approximately the same value at a point. So they are close. This is similar to the use of the sandwich lemma from Calculus I to say two function have the limit. However, these arguments say nothing about the existence of a derivative at the point. For that, we have to do more work. The better measures of distance take into account measures of whether the curves are not only close in distance but also close in tangency.

There are gradations of closeness. Closeness in distance is the coarsest measure of closeness. The next level is closeness in distance and tangency. The next level of closeness needs to compare the second derivatives of the curves or the curvature of the curve at each point. For each derivative, there is a corresponding notion of closeness. We will not go beyond second order closeness in our considerations. But, you need to be aware of other measures of closeness.

In particular, all of the measures mentioned so far are pointwise notions of closeness. None of them consider whether or not the models have the same length or size. For instance, one could argue that the model of the circle in Figure 5 is bad not only because of the tangents are bad, but because the length of the curves are wildly different. To define these more accurate measures of closeness requires considering the calculus of parametric curves and other notions of distance.

**Exercies**

1. Calculate the distance between the circle $\alpha(t) = [\cos(t), \sin(t)]$ with $0 \leq t \leq 2\pi$ and a wavy circle $\beta(s) = [(1 + 0.1\sin(3s))\cos(s), (1 + 0.1\sin(3s))\sin(s)]$ with $0 \leq s \leq 2\pi$.

2. Calculus the distance between the line segment $[0, t]$ with $0 \leq t \leq 1$ and the parabola $[s^2 + s + 1, (4s - 1)/2]$ with $0 \leq s \leq 1$.

3. Write the word ROSE in lower-case script letters.
   (a) What is the minimum nunmber of points you need to obtain a reasonable approximation of the letter R using piecewise linear curves?
   (b) How many points do you need to get a good approximation of the letter R using piecewise linear curves?
   (c) Approximate the word ROSE using piecewise linear curves and roughly 5 points per letter. How good of an approximation do you get?
4. Write the word MATH in lower-case script letters.
   (a) Approximate the letter M using a nonsmooth piecewise circular curve. How many points did you use? How did you decide which points to use?
   (b) Approximate the letter M using a smooth piecewise circular curve. How many points did you use? How did you decide which points to use?
   (c) Approximate the word MATH using piecewise circular curves and roughly 5 points per letter. How good of an approximation do you get?
5. Write your initials using both piecewise linear curves and piecewise circular curves. Which is the better aproximation to your initials?