MA 323 Geometric Modelling
Course Notes: Day 02
Model Construction Problem

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In the next few days, we will introduce some of the basic problems in geometric modelling, and some very simple methods for solving them. Each of the motivating problems has different flavor depending on the type of object that is being created. For our purposes, there are two types of objects to model; curves and surfaces. One can consider solids as a different class of objects, but since a solid is bounded by a surface or a collection of surfaces, for our purposes in this course a solid is represented by its bounding surfaces. In this chapter, we will pose the motivating problems in generic terms, but concentrate on the version of the problems for curves when we discuss the solution techniques. The generalization of the motivating problems to surfaces is straightforward and simple, but the solution methods are more complex to state in mathematical terms.

2.1 The Model Construction Problem

The first and most basic problem in geometric modelling is to construct a model of an object by specifying geometric primitives. In this problem, the modeller needs to supply *simple geometric information* (points, lines, planes, etc) in order to obtain a model of the object. The solution method takes the geometric primitives and constructs the model. The solution method is normally algorithmic and can be modified at the modeller's discretion, depending on the exact problem that needs to be solved.

The solution methods to the model construction problem generally are described by the type of objects that they generate. For instance, in the subsections that follow, we describe two types of curve constructions; piecewise linear curves and piecewise circular curves. There are of course other curve construction methods that are substantially more complicated to describe. In general, these more complicated methods are better modelling methods. We will discuss some of these other methods later in the course, but piecewise linear curves and piecewise circular curves suffice for a discussion of the construction problem and some of the important issues in the construction problem.

It is worth repeating that the solution method to the construction problem greatly affects the solutions to any other problem that the modeller may want to tackle. This is due to the fact that construction problem specifies the type of model and therefore the properties of the model. The properties the model possesses will of course restrict the ability of the model to solve other problems and affect the analysis of the model. Sometimes the model's ability to solve the problem is hampered by the construction method. In fact, one practical concern that we will not discuss that often is the conversion of one model to another model.

2.2 Solution by Piecewise Linear Curves

The simplest solution to the model construction problem for curves is to construct a piecewise linear curve to approximate a curve. A piecewise linear curve is just a finite sequence of line segments. To specify a piecewise linear curve, one only needs to specify a sequence of points $\{\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_m\}$, and then form the line segments $\mathbf{p}_i\mathbf{p}_{i+1}$. Thus, we form m line segments from the $m+1$ points. This provides a very simple geometric construction to approximate a drawn curve in a plane, see figure below. The actual curve constructed in this manner is sometimes called a *polyline* and the function that describes the curve is a piecewise linear interpolant or piecewise linear function.

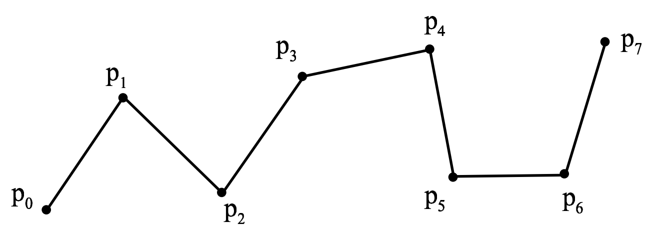

Figure 1: A piecewise linear curve

It is a basic but deep result in mathematical analysis that any continuous curve can be approximated closely in distance arbitrarily well with a piecewise linear curve. This is a direct result of the ϵ-δ definition of continuity. A significant problem with this basic result is that it does not say how to arrange the points and even how many points are needed. It simply says that it is possible to approximate any continuous curve with a *piecewise linear curve*. Nevertheless, piecewise linear curves and piecewise linear function are the fundamental building blocks of every graphics engine and every approximation scheme. The reason for this is simple. The pixels on the screen can be viewed as points, and to create a curve, we need to it is easy to sample points on the curve and connect them by straight lines. Moreover, digital computers can only add and multiply, which are the fundamental operations of linear approximations.

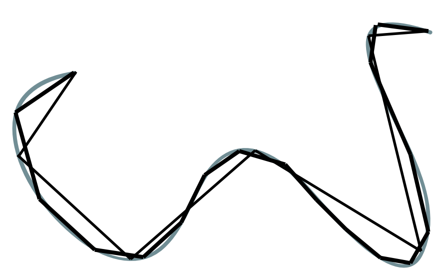

Figure 2: Increasing accuracy of approximation with more data points

There are several obvious problems with using piecewise linear curves to provide a good solution to the construction problem. A serious problem is that to obtain a good approximation with a piecewise linear curves one might need to supply a large number of points, and consequently a large number of line segments. A second problem is that the model generated by piecewise linear curves is not smooth. There is a noticeable angle at each point, when only a few control points are chosen. When more points are used the angle becomes less discernable, but this only highlights the serious problem of obtaining a good approximation.

It is worth mentioning that despite the above problems with piecewise linear curves, they are an important tool in geometric modelling. One reason for this is that most of the other methods will eventually use a piecewise linear curve using a high number of points as an approximation of the model curve that it will generate. The modeller does not have to generate all the points themselves, the modelling method typical generates most of the points from a few select points generated by the modeller. The reason for this is simple. The methods for generating geometric models are limited by the discrete nature of computers and the resolution of the screen and computer control. To connect between discrete points that are very close together, most computer systems will use a straight line for motion and as most problems are continuous and differentiable, calculus ensures that locally curves can be approximated by straight lines.

2.3 Solution by Piecewise Circular Curves

An alternate solution method to piecewise linear curves that can remove the smoothness problem in the model curve is to use piecewise circular curves. The difference between piecewise circular curves and piecewise linear curves is to replace the line segments with circular arcs. This will be a fundamental tool in geometric modelling to use basic curve elements, and joining them together to create a model.

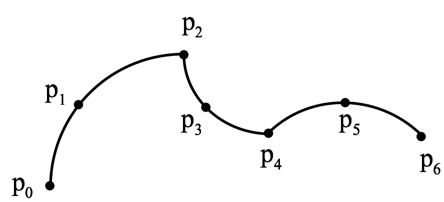

Figure 3: A nonsmooth piecewise circular curve

To construct a piecewise circular curve, recall that it is possible to construct a circular arc that passes through three noncollinear points, see below for the construction. Therefore, by providing a sequence of points $\{\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_n\}$ with $n = 2m$ an even number, one can construct a circular curve by finding the circle that passes through each set of three points $\mathbf{p}_{2k}$, $\mathbf{p}_{2k+1}$ and $\mathbf{p}_{2k+2}$ with $k = 0, 1, \cdots, n-1$ and then finding the circular arc that goes from $\mathbf{p}_{2k}$ to $\mathbf{p}_{2k+2}$ and passes through $\mathbf{p}_{2k+1}$, see diagram above. We note it is not necessary to require that the points $\mathbf{p}_{2k}$, $\mathbf{p}_{2k+1}$ and $\mathbf{p}_{2k+2}$ to be noncollinear. The case of three collinear points is covered by noticing in the construction of a circle through three noncollinear points as the the points get closer to being collinear the circle produced approaches a straight line. Therefore, for three collinear points the circle through the three points is a straight line. The circular arc is then the line segment $\mathbf{p}_{2k}\mathbf{p}_{2k+2}$. Note the circular arc "makes sense" (at least classically) when $\mathbf{p}_{2k+1}$ is between, $\mathbf{p}_{2k}$ and $\mathbf{p}_{2k+2}$, though we will see later that it makes sense projectively when $\mathbf{p}_{2k+1}$ is not between $\mathbf{p}_{2k}$ and $\mathbf{p}_{2k+2}$.

GEOMETRIC CONSTRUCTION: *The construction of a circle through three noncollinear points.* Let A, B, C be three given noncollinear points. To construct a circle through these three points, we first construct the triangle with vertices A, B, C. Then we construct the circumscribing circle for this triangle. This is done by first finding the midpoints of the sides of the triangle, called these A', B', C' with A' on side BC, B' on side AC and C' on side AB. Construct lines perpendicular to AB through C', perpendicular to AC through B', and perpendicular to BC through A'. These lines all intersect at a common point Q called the circumcenter of the triangle. This point Q is the center of the circumscribed circle to the triangle ABC. To see this note that the Pythagorean theorem applied to the right triangles $AC'Q$ and $BC'Q$ (since $C'Q$ is in both triangles and $dist(AC') = dist(BC')$ shows that the distance $dist(AQ) = dist(BQ)$. Applying the same argument to the pair of triangles $BA'Q$ and $CA'Q$ yields that $dist(BQ) = dist(CQ)$. Therefore the circle of radius $dist(AQ)$ through Q passes through A, B, C.

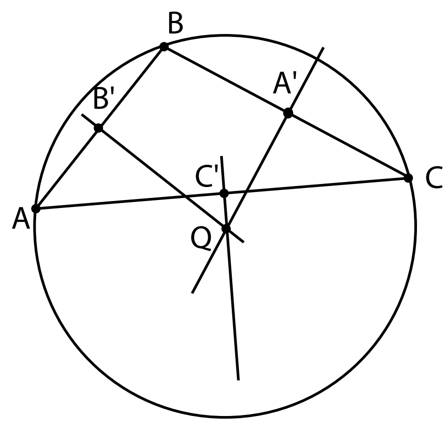

Figure 4: Construction of a Circle Through Three Points

It is not hard to modify this construction of a circle through three noncollinear points, to obtain a circular arc that starts at point A and ends at point C and includes point B. All that one needs to do is construct the lines AB and BC, then construct the midpoints and the perpendicular bisectors of these lines. The point of intersubsection is the center of the circlular arc. Then draw the arc.

In the construction of a (nonsmooth) piecewise circular curve above, we construct m circular arcs from the $2m + 1$ given points. We note this construction only generates a continuous curve. At the points $\mathbf{p}_{2i}$ with $i = 1, 2, \cdots, m - 1$ there is generically a discernible angle. [Generically is the mathematically terminology that means usually or more accurately unless unusual circumstances prevail.] This means that without quanitifying measures that if one randomly chooses the points $\mathbf{p}_i$ there will be a discernible angle almost always at the points $\mathbf{p}_{2i}$. This is not aesthetically pleasing or really practical. We have only replaced the straight lines with circles, at the inconvenience of having to supply more points.

The advantage of piecewise circular curves is that one may with the appropriate choice of the points $\{\mathbf{p}_i\}$ obtain a smooth model. It is important that the appropriate choice of the points follow simple rules that is easy to compute. To develop these rules, we make the following observations.

- For a piecewise circular curve to be smooth, we need the tangent lines at the *joint points* $\mathbf{p}_{2k}$ $(k = 1, 2, \cdots, m-1)$ as computed from each circular arc to be identical.

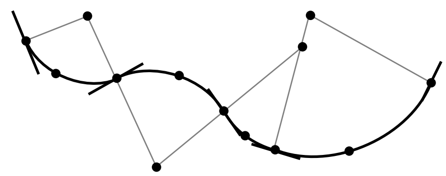

Figure 5: Illustration of Properties of Smooth Piecewise Curves

- The fact that the tangent line to a circle is perpendicular to the radial line implies that radial lines (line between the center and a point on the circle) at $\mathbf{p}_{2k}$ must be the same for each circular arc. This means that centers of consecutive circular arcs must be collinear with the *joint point* that the arcs share in common.

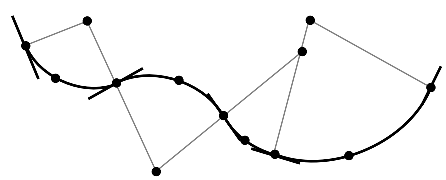

Figure 6: Illustration of Properties of Smooth Piecewise Curves

- It is possible to construct a piecewise circular curve consisting of two circular arcs by giving the three points $\mathbf{p}_0$, $\mathbf{p}_2$ and $\mathbf{p}_4$ and the tangent line at $\mathbf{p}_0$. (see ASIDE: *Construction of circle from two points A and B and a tangent line l at A*)

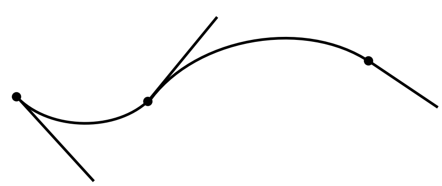

Figure 7: Construction with a Tangent Line

- Generalizing the observation, one can construct a smooth circular arc consisting of m arcs by giving $m+1$ points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, $\cdots$, $\mathbf{p}_m$ and a tangent line $\mathbf{l}$ at $\mathbf{p}_0$. First construct the circular arc starting at $\mathbf{p}_0$ and ending at $\mathbf{p}_1$ using the tangent line $\mathbf{l}$. Construct the tangent line at $\mathbf{p}_1$ from this first arc, and use that tangent line to construct the circular arc from $\mathbf{p}_1$ to $\mathbf{p}_2$.

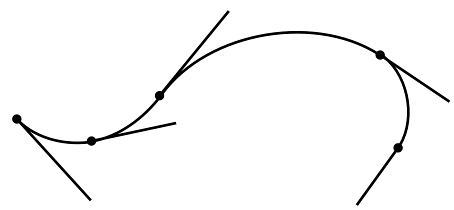

Figure 8: Construction of a Smooth Circular Curve

The basic construction that allows one to construct a smooth piecewise circular curve requires a slight modification of the construction of a circle through three noncollinear points. In this construction, we want to construct a circle given two points and a tangent line at a given line. Then, a slight modification of the construction will give the desired circular arc.

GEOMETRIC CONSTRUCTION: *Construction of circle from two points A and B and a tangent line l at A*: In this construction, there are two cases to consider. The first case is when B lies on the line l. In this case, the circular arc is the line segment AB. The second case is when B does not lie on the line l. The circle is constructed as follows.

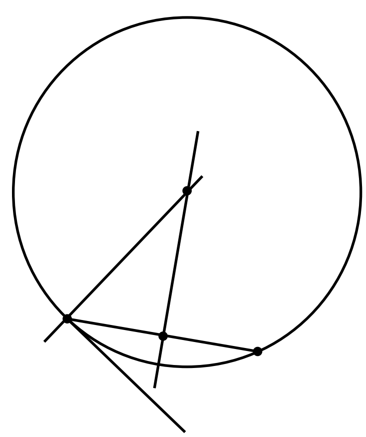

Figure 9: Construction of a Circle from Two Points and Tangent Line

- Construct a line l' perpendicular to l passing through A. (This is the radial line of the circular arcs that pass through A).

- Construct the perpendicular bisector m of the line segment AB.
- Let C be the intersubsection of the lines l' and m. This is the center of the circular arc. The lengths AC and BC are equal because of the Pythagorean theorem.

To construct a circular arc, given a tangent line l at A, one proceeds as above to construct the circle, but in addition one must choose or be given a direction on the line (a tangent vector at A). This allows to choose which arc of the circle goes from A to B. We choose the arc leaving the point A going towards the point B in the direction chosen. Notice that for the collinear case, the only orientation that makes sense classically is the direction that points from A towards B. However, in either case, there is an implied tangent vector for the circular arc at point B given by the vector heading in the direction that the arc continues, see diagram below. This allows us to continue the construction from B by adding a new circular arc, and thus obtaining a smooth circular curve.

The construction of a smooth piecewise circular curve using the construction of a circle from two points and a tangent line allows us to obtain a globally smooth curve. Notice that for this construction, we need to supply $m+1$ points (the end points of each circular arc) and a tangent line at one of the points, and moreover given any $m+1$ points there is a one-parameter family (defined by a choice of a tangent line) of smooth piecewise circular curves. An alternate method is to use the first three consecutive points (or any three consecutive points) to define a circular arc and then base the rest of the circular constructions off this circle, as the construction of one circle defines the the necessary tangent line for the remainder of the circles.

Exercises

1. (Computational) Approximate the curve below with a piecewise linear curve using four line segments, five line segments, ten line segments. How many line segments are needed to produce a good approximation? How should we place the "control points" determining the line segments?

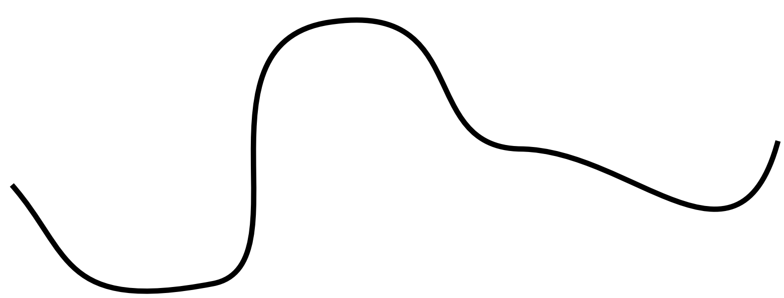

Figure 10: Approximate this Curve with a Piecewise Linear Curve

2. (Thought) Does the construction for a piecewise linear curve require the points to be in a plane? Can you construct a piecewise linear curve with the "control points" in three dimensional space? What needs to be modified from the planar case?

3. (Computational) Approximate the curve below with a smooth piecewise circular curve with four arcs, five arcs, ten arcs. Is it possible to produce a good approximation with a piecewise circular curve? (Why or why not?) If so how many circular arcs are needed to produce a good approximation? How should we place the "control points" determining the circular arcs?

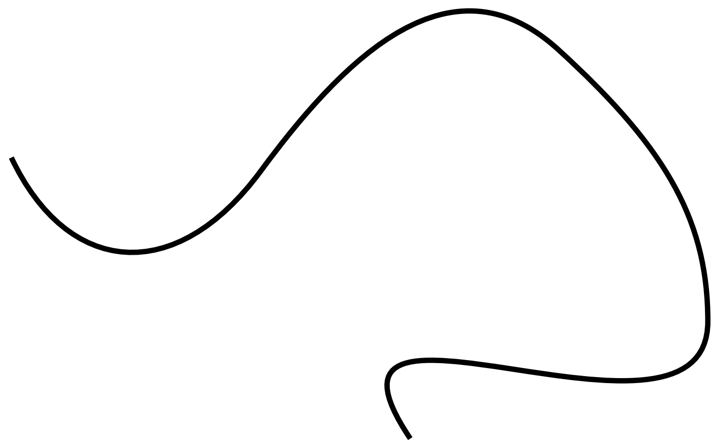

Figure 11: Approximate this Curve with a Piecewise Circular Curve

4. (Interactive) Complete the interactive exercises associated with the applet on the course web-page Piecewise Circular Curves

5. (Thought) Does the construction for a piecewise circular curve require the points to be in a plane? Can you construct a piecewise circular curve with "control points" in three-dimensional space? What if anything needs to be modified from the planar case?