MA 323 Geometric Modelling
Course Notes: Day 01
Course Introduction

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The subject of geometric modelling is a blend of different types of geometry, some mathematical modelling, some numerical analysis, and discipline specific applications. In these course notes, we will concentrate of the subdiscipline of geometric modelling known as computer-aided geometric design, which was formed from the mathematical structures and methods used in CAD/CAM systems and subsequently exploited in computer graphics and computer animation. Other subdisciplines of geometric modelling include solid modelling and the recent discipline of modelling with modelling by solving (geometric) partial differential equations and solving geometric variational problems.

Like most applied mathematics disciplines, geometric modelling arose as a collection of related techniques used to solve specific problems. The problems motivating geometric modelling came directly from industrial changes in the automobile industry. the aerospace industry, and the ship-building industry, with the development of computers and their use in the design and manufacturing of planes, trains, automobiles and ships. These changes seem to have come first in the automobile industry and the aerospace industry in the late 1940's and early 1950's. Now, with the increased use of computers in everyday life, geometric modelling methods are pervasive, and seen everyday through the use of computer graphics and computer animation. In fact, the field of geometric modelling continues to develop with the development of new and better methods and improvements in technology. In addition to the application and modification of the original methods to other fields, most notably computer graphics, imaging science (computer vision and related fields including bio-medical imaging), and robotics.

The goal in this course is to examine the basic underlying geometric structures that are used in solving some problems in geometric modelling. This will involve a brief introduction to several areas in geometry and how each area of geometry can be used in solving problems arising in geometric modelling. We emphasize that the point of this course is to introduce various geometric structures and apply them to practical problems using geometric reasoning. The reader will find few formal rigorous proofs in these notes, but several informal proofs that emphasize geometric reasoning.

In the remainder of this today's notes, we discuss the general subject of geometric modelling in more detail, the art of mathematical modelling, some of the basic mathematics involved in geometric modelling, and some historical background in the development of geometric modelling and computer-aided geometric design. The course develops by first introducing the different types of motivating problems that we will consider, and some simple (naive) solution methods using very basic geometric structures. We then develop more sophisticated geometric structures that will be used in the remainder of the course, and apply these structures to solve the motivating problems.

1.1 What is geometric modelling?

Geometric modelling is a subdiscipline of the more general subject of mathematical modelling. The emphasis in geometric modelling is on building a "geometric model" of an object in order to describe its shape and geometric properties. The object could be a real physical object such as the body of a car, the hull of a ship, or a chair. The object could also be a virtual object such as a character in a movie like "Buzz Lightyear" in *TOY STORY* or "Yoda" in *STAR WARS: A Phantom Menace*. Moreover, it could be a *hypothetical* object that is obtained by a geometric metaphor as an image of brain activity in an MRI scan. Whatever the object, the aim in geometric modelling is to create a *mathematical model* that describes and represent the *shape* of the object, that is the *geometric properties* of the object.

In this course, the exact reason for constructing the model will be viewed for the most part as irrelevant when discussing the various methods that are used to construct a geometric model. However, the reason does become important when deciding the *goodness* of the model. It is the discipline specific application that will determine what factors are irrelevant in testing the *goodness* of the model. For example, when designing an aircraft, an engineer may want to construct a geometric model in order to analyze the behavior of the airflow over the wing to reduce the turbulence felt by the passengers. In this situation, the shape of the wing determines the drag coefficient and thus affects the airflow, but the exact determination of the drag coefficient is effected subtlety by the shape of the wing and the rest of the plane, and thus is not easy to predict from the shape. As a second example consider the construction of a model of a human heart that is used to analyze medical conditions by a nonintrusive medical scan, that is imaging technology or a probes. The construction of a geometric model of the heart is important as the geometry of the model will effect the fluid flow in the heart and the interpretation of the signals received by the probe or the scanning technology. However, the effect of changes in the geometric model are hard to predict, and the interpretation of the data is probably very sensitive to differences in the geometry. These two examples show that there are typically other factors that are involved in the construction of the geometric model in applications.

The other factors shown by the above two examples are extremely important in analyzing the goodness of a geometric model. For the most part, we will leave the analysis of a geometric model to the various application domains that use geometric modelling as a tool. In this course, we will only concern ourselves with analyzing the goodness of a geometric model subject to aesthetic constraints and geometric properties, and therefore we will use aesthetic judgement and mathematical measures of aesthetic goodness by geometric properties to test whether or not a model is good or bad. Frequently, our mathematical measures will be phrased as an optimization problem, such as minimize the length of the curve that described the object subject to the curve satisfying certain constraints. This gives the general flavor of the analysis in the various application domains without introducing unneeded terminology and extraneous information from the application domain areas.

For our purposes, the emphasis in geometric modelling on describing the *shape* of an object, and in this course on modelling the aesthetic properties of the shape is the difference between the more general field of mathematical modelling and the specific discipline of geometric modelling. The field of mathematical modelling usually concerns the mathematical representation of a *physical phenomenon* (or man-made phenomenon), such as one finds in the sciences and engineering disciplines. Some would argue that the sciences and engineering disciplines are only specific applications of mathematical modelling, as the art of mathematical modelling consists more of a methodology than a coherent set of axioms and mathematical structures, much more like the sciences and engineering disciplines. There are general mathematical methods employed in mathematical modelling, but the methods are only loosely connected (see the section on *The Art of Mathematical Modelling*).

Geometric modelling follows the same methodology as mathematical modelling, but the restriction to shape issues (geometric properties) provides geometric modelling with a more coherent structure. Specifically, the restriction to geometric properties means the majority of the methods involve the mathematical subjects associated with geometry and the associated numerical algorithms. Depending on the fields of geometry emphasized, geometric modelling has different flavors, dominated by two subdisciplines computer-aided geometric design (CAGD) and constructive solid modelling (CSG). There are other methods employed in geometric modelling include modelling with (geometric) partial differential equations and the use of calculus of variations, statistical shape modelling, wavelet methods, implicit surface modelling, et cetera. In this course, we will mainly concern ourselves with the methods in CAGD that are based on analytical geometry, differential geometry, projective geometry, combinatorial geometry and topology. The methods associated with CSG concern more boolean algebra and set theoretical operations, which require a substantially different mode of thought. In solid modelling, one defines geometric primitives and then uses boolean algebra to add and subtract these primitives to form more complex objects. However, the methods in CAGD still play a role in CSG as the methods in CAGD allow one to join, blend and smooth the primitives in CSG to create even more complex objects.

In geometric modelling, there are two basic types of problems, model creation and model analysis. Model creation is the primary problem. This problem involves developing methods for constructing a model from geometric primitives and geometric properties, and thus answers the general aim of geometric modelling. The model analysis problem is in some sense the more important problem for it determines whether or not one has constructed a good model. It is worth mentioning that these two types of problems are often interrelated in practice. The solution to the model creation problem will affect the analysis of the model, and the desired analysis of the model will affect the construction of the model. This is because the choice of a modelling method will determine how the model can be analyzed and how the model needs to be analyzed will affect the choice of a modelling method.

Before describing some of the methodology in geometric modelling, it is important to understand that geometric modelling is both an old subject and a relative young subject. The practical geometry of artists and builders that was developed at the start of civilization was geometric modelling. Much of the early history of geometry is geometric modelling, how to construct lengths and double a cube, trisect an angle, etc. These were modelling problems that were posed by design problems. Formal geometry became an abstract art for development of reasoning with the ancient Greek civilization. Geometric modelling then was relegated to artistic endeavors; artists designing sculptures and architects designing buildings. When an artist or architect builds a model in clay before working with stone in full-scale, they are working with a geometric model. The methods used in these situations were more artistic rather than scientific.

Modern methods in geometric modelling are scientific based and depend on mathematics. Instead of designing with clay or other cheap materials, modern geometric models are constructing via computers. In fact, the resurgence of geometric modelling is primarily a result of the modern age where computers revolutionized industrialization. Almost everything produced in the modern age is produced from automated assembly lines, where computers control the manufacturing. The computers must be told how to assemble and cut the product. This requires a geometric understanding of the shape of the object that can be conveyed to the computer. This means that when the object is designed it must be designed in such a manner that the computer can understand its instruction.

Computers rely on mathematical structure to perform operations, and because of this geometric modelling had to become more of a mathematical discipline. In fact, much the development of geometric modelling occurred during the modern industrialization age from the 1940's to the 1980's. During this time, the development of the methods in geometric modelling were at times carefully guarded industrial secrets, and thus many of the methods were discovered several times and cast in described in different manners. The majority of the methods that are presented in this course were developed by engineers and mathematicians working for various companies in the automobile industry, the aerospace industry, and the ship-building industry that pioneered the modern industrialization.

During the same time, the field of computer graphics was in its infancy but making rapid strides. In part, the quick progress made by computer graphics was due to the rapid advance of CAD/CAM systems. However, since the 1970's computer graphics has become an established field and has started developing its own identity separate from CAD/CAM and CAGD. However, there is still a close connection between computer graphics and geometric modelling, with many practitioners of geometric modelling working in the computer graphics industry and many innovators in the computer graphics industry working on problems in geometric modelling. For example, most of the special effects in movies that are accomplished using computers require the use of some of the methods in geometric modelling to create the visual effects experienced in the movie. These same methods are used in computer vision, computer graphics, and computer games to create a sense of realism and the 3D effects that mimic reality.

The methods in geometric modelling also arise in other applications. They are employed in numerical solutions to most engineering problems involving partial differential equations and calculus of variations (optimization problems for functions - problems of shape optimization is a current active research problem). The reason for this is that the methods in geometric modelling allow one to discretize a continuous problem. The discrete version is usually more amenable to computation and simulation than the continuous version. In fact, the finite element method typically employs many of the same methods as used in geometric modelling to build the solution. In addition, the methods of geometric modelling can also be applied to robot control and manipulation and robot vision. This is because geometric modelling allows a computer a method for describing geometric properties of space and the location of objects which fundamental to the control of a robot.

1.2 The Art of Mathematical Modelling

As stated in the opening of the previous section, the aim of geometric modelling is to create a *mathematical model* that represents the shape of a physical or virtual object. Before proceeding further, it should be noted that the model is not necessarily the object. The model should behave in the same manner as the object up to some well-defined tolerance. This tolerance is a measure of the *goodness* of the model as an approximation to the object. However, the model can be an exact representation of the object if the object is constructed from the exact data used to construct the model. This means if the object is created from the model. But typically, one creates a mathematical model to be able to analyze the behavior of the *object modelled* by analyzing the behavior of the model. With this stated, the advantage of the mathematical model over the object being modelled is that the model should be easier to analyze than the object. For instance, it is normally easier to run a computer simulation than to construct the object and run physical tests, at least in the initial development phase.

The creation of a mathematical model typically follows the scientific method, where one first develops a hypothesis (the model), and then test the hypothesis before drawing conclusions based on the hypothesis. This means the first step in creating a mathematical model is to obtain a precise description of what is to be modelled. This means defining the important features, and converting these important features into mathematical terms. One also needs to consider what type of mathematical analysis will be used and how one will assess the accuracy of the model. As the important features are precisely described, one might needs to conduct experiments or computer simulations to compare the model to its real-life counterpart. During the comparison, one needs to determine whether the model is a good model (it accurately reflects the object's important features) or a bad model (it does not accurately reflect the object's important features). Then, one needs to determine whether to refine the model or alter the model or construct an entirely new model. Once a fairly good mathematical model has been constructed, one can then use the model to make predictions and conclusions about the physical phenomena based on the behavior observed in the model.

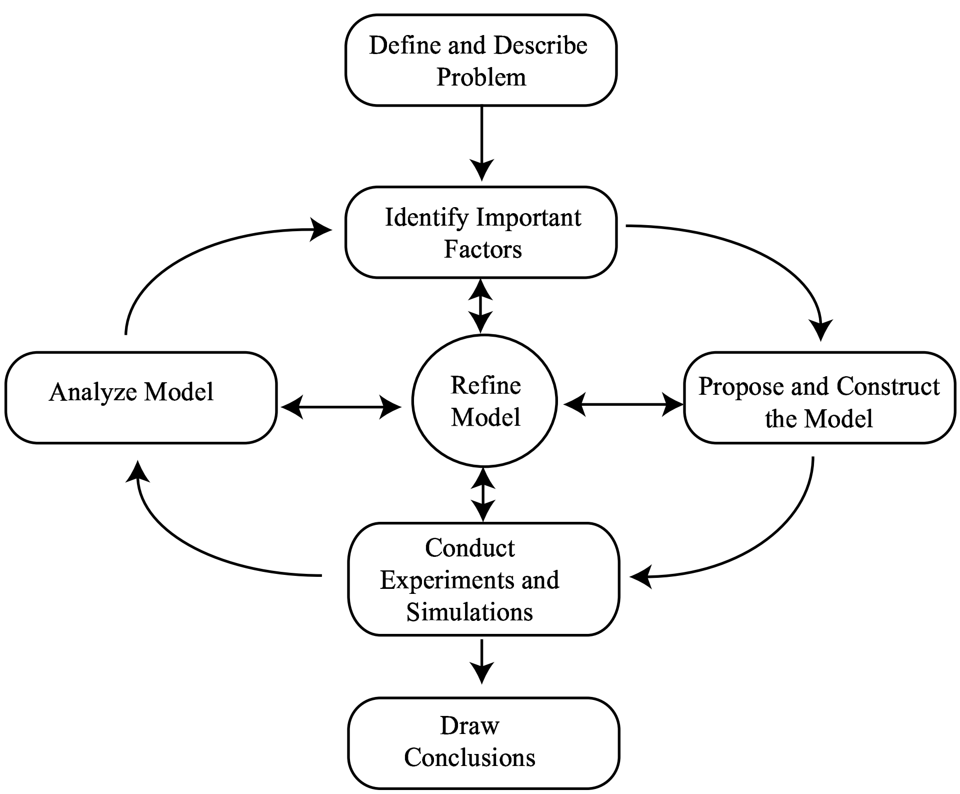

Figure 1: A Flow Chart for Mathematical Modelling

The method outlined above can be described by steps below with a graphical description given by the flow-chart in Figure 1.

- Develop a clear and concise description of the problem to be examined.
- Identify the important factors that affect this problem or may play a role in the solution.
- Propose a mathematical model for the problem.
  - Decide what type of mathematical analysis is appropriate for the problem.
  - State any equations that should relate the important factors or describe factors associated with the problem.
  - Make simplifying assumptions.
  - Perform necessary mathematical analysis to obtain solution.
- Conduct experiments or simulations.
- Analyze the model and decide whether it accurately reflects the phenomena.
- Refine or alter the model as appropriate to improve accuracy of the model.
- Draw conclusions about the phenomena based on the model.

Throughout this process, one should be continually trying to improve the model (or scientific hypothesis). Typically, one starts with a simple model and adds to the complexity of the model as needed to obtain greater accuracy. In this approach, it may be necessary to start again and redefine the problem and construct a completely new model.

In geometric modelling, we follow the same general procedure. The difference occurs in the minutia of each step. For instance, when describing the problem and the important features, in geometric modelling they should all be shape related. For instance, the features may restrict the size of the object by saying the object fits within a box with specified length, width and depth or state the object is radial symmetric with a specified axis. These specifications have the effect of reducing the choice for creating the model to choosing the method for creating the model. The choice of method for creating the model also specifies the method for analyzing the closeness between the model and the object, as each method normally has a preferred method for measuring the distance between objects. This means that one does not have to choose between different types of mathematical analysis to analyze the model one only has to choose the method for creating the model and the method for measuring the closeness of the model (the physical measurements by which one is to judge the measure of closeness), which means that instead of conducting experiments and simulations one needs to only perform the necessary measurements. Thus, the restriction to geometric modelling considerably simplifies the process of creating the model by focusing the type of model and the methods that can be employed.

It should be mentioned that if the geometric model is to be used in another mathematical model or in a simulation this other problem can influence the method to be used in creating the geometric model. For instance, in analyzing the airflow over a wing (a fluid dynamics problem), an engineer will want to create a geometric model of the wing. However, the methods that he must use to create the model might be dictated by the method by which he will analyze the fluid dynamic problem. This may affect the choice of the measure of closeness used, and the exact nature of the geometric modelling problem. In this course, we will thus focus on geometric models mainly in the aesthetic sense not in the practical engineering sense for simplicity. The caveat is in practice to consider the reason for creating the model before choosing the method for constructing the geometric model.

1.3 Mathematical methods in geometric modelling

Mathematically, the field of geometric modelling consists of a loosely connected set of methods that can be used to describe the shape of an object and that can be used to create an approximation of an object. In creating and analyzing geometric models, there are no rules only traditional guidelines on when to use each method and which method is best in certain situations. For these reasons, geometric modelling like mathematical modelling is in some sense more like an art than a science. Some of this view stems from the goal of geometric modelling and mathematic modelling of constructing an accurate or "best" representation of an object. This is completely different from the general goal of a science where one is seeking general truths or fundamental laws. In fact, as we shall see, in the next section, some of the fundamental mathematical structure that will used in this course originates with artists. Best can be quantified once a measure of closeness is chosen, and then it is an optimization problem to minimize the error. For the most part, we will not seek the optimal answer but rather a good answer.

The methods and tools that are used in geometric modelling come from every branch of modern mathematics. We will emphasize the methods that are mainly derived from Euclidean geometry, analytic geometry and basic differential geometry (calculus based geometry). However, we shall also use some methods from discrete mathematics, linear algebra, Boolean algebra, topology and numerical analysis. No specialized knowledge beyond a firm grasp of multi-variate calculus (and its pre-requisites) is required to read these notes.

We repeat the main emphasis in this course is on the subfield of geometric modelling known as computer-aided geometric design (CAGD), which is closely related to computer-aided design and computer-aided manufacturing (CAD/CAM) and computer graphics. The other main subfield of geometric modelling is constructive solid modelling (CSG). Solid modelling is also used in both CAD/CAM and computer graphics. However, it requires a different mode of thought, more Boolean algebra (the addition and subtraction of basic geometric objects), and then uses some aspects of CAGD to join these basic objects. There are other subfields of geometric modelling that employ different mathematical methods, but such methods will not be mentioned in further detail in this course.

We will emphasize basic methods and geometric concepts in this course. There are more sophisticated methods that employ the same basic ideas, but require more sophisticated mathematical algorithms and background. For instance, when discussing Bezier curve and de Casteljau's algorithm we will not use blossoming in detail. The reader should be aware there are other methods that can be employed and more sophisticated data structures and algorithms. The approach taken in this course and in these notes is to introduce the main ideas and introduce the concepts first, and then time permitting go into the sophisticated details.

The main mathematic tools that we will use in this course are from analytic geometry, affine geometry, projective geometry, differential geometry, and linear algebra. We concentrate on the methods that stem from analytic geometry, differential geometry and linear algebra (vectors and matrices), because these methods pull upon the same foundations as calculus, especially calculus applied to vector valued functions. We will introduce and use some of the main concepts in affine geometry, projective geometry, combinatorial geometry and topology, but the main emphasis in this course is with the calculus based methods. Throughout the course, one common element that the reader will find in most of the methods is an emphasis on the difference between numerical computation and efficient numerical computation. In the beginning of the course, we consider simple computational methods, not necessarily efficient computational methods. For instance, the basic methods we will present in the next few days are not numerically efficient, but they are simple to explain and highlight the general aims in geometric modelling using only basic mathematical terminology. We will later present the more sophisticated mathematical ideas and more efficient algorithms.

1.4 Some Historical Background

The mathematics involved in describing the shape of an object and its properties goes back to ancient times, i.e. Euclidean geometry. Originally, Euclidean geometry was developed by the ancient Egyptians and Babylonians for measuring and describing fields and buildings. It was not until the ancient Greeks that the subject developed its formal logical structure as a training tool for the mind in order to master the art of rhetoric and debate. Our concern with Euclidean geometry will resemble more practical concerns than the theoretical development of the subject admired by the ancient Greeks. It should be noted that practical Euclidean geometry is mostly a dead subject in mathematical circles, though it is still one of the main mathematical tools used by carpenters, craftsmen and artisans.

The ancient Greeks and their mathematical predecessors also studied some curves and surfaces, though they were limited in the type of curves and surfaces that they could study. This is partly because they mainly limited themselves to curves and surfaces that could be constructed by straight-edge and compass. They did allow the use of conic sections (the intersection of a plane with a cone) for some constructions. In fact, the conic sections were a tremendous source of mathematical interest to the mathematical descendants of Euclid and his predecessors with entire books devoted to their study. The ancients also considered some types of mechanical curves and surfaces, meaning curves and surfaces that could be describe by the moving circles and lines.

Still with the limitation to straight lines and circles, great works of architecture and building could still be done. The beginnings of the use of curves made from straight lines and circles appears to be for the purposes of shipbuilding. Ships were standardly built from reusable templates, that were typically made by joining arcs of circles together and varying the sizes of the arcs along the keel. Records of this method of ship-construction began in the early days of the Roman empire, but may have begun earlier. Drafting and designing the hulls of a ships was done much later in the 1600's in the British empire. By which time, the construction of ships was done by more mechanical means; mechanical splines that bent the planks used in the hull according to the ship-builders desires.

This limitation on the types of curves and surfaces that could be used for mathematical design continued for a long time. This limited the use of mathematics in geometric modelling. The turning point occurred during the Renaissance and the Scientific Revolution that followed. During the Renaissance, artists (Albrect Dürer, Leonardo da Vinci, and others) studied what they called perspective geometry in order to more accurately represent three dimensional objects on a two-dimensional canvas, though this subject was also of concern to Euclid under the name of Optics. This field would later be systematically studied and explored by mathematicians in the 17th, 18th and 19th centuries and called projective geometry. However, it was during the 17th century that the types of curves and surfaces that could be studied were substantially enlarged. During this time, the separate mathematical disciplines of algebra and geometry were united by Fermat and Descartes. In fact, soon after the development of analytic geometry by Fermat and Descartes, calculus in its modern form was developed independently by Newton and Liebnitz.

With the invention of calculus, one could study arbitrary curves and surfaces, since it was now possible to describe a curve or a surface by its geometric or physical properties. In fact, one of the major impetuses in the development of mathematics and calculus in particular has always been the delightful interplay between geometry and physics. Immediately following the invention of calculus, the above impetus led to a rapid succession of applications of calculus and the solution of many problems in science and engineering with the development of new methods for investigating mathematical problems in physics and geometry. The entire discipline of differential geometry (the study of curves and surfaces through calculus) grew during the 18th and 19th centuries from practical applications of calculus to problems in measuring the earth.

Throughout this time, the subject of geometric modelling lay essential dormant. There was some work done by Gaspard Monge and his school on descriptive geometry, which laid the foundations for constructing engineering drawings. There was also some work done in terms of physical constructions through tensions and strains for creating hulls of ships with wooden beams, by using mechanical splines. The construction of splines originally was done through actually physical means. But, for the most part, the subject of geometric modelling was waiting for the development of the technological tools which would enable problems to be solved efficiently. The development of computers in the 1940's and 1950's and their use in design and manufacturing prompted interest in geometric modelling in its modern form. The use of computers to describe the shape of objects required using the mathematics and geometry that was developed throughout the ages to input and display objects in a realistic manner.

Key developments occurred during World War II. For instance, while working for North American Aviation (a company that built many fighter planes for the US Army and the US Navy), R. Liming wrote a book entitled *"Analytical Geometry with Application to Aircraft"*. This book combined classical drafting methods with computational techniques using conic sections. The main influence of this book was to change the drafting techniques from manually drawn curves to storing numbers. This reduced ambiguities in the drawings and left no room for individual interpretation of the plans.

Other advances then occurred rapidly with the advent of computers. The use of computers for controlling the milling machines used to create automobiles, planes, trains and ships required their instructions in terms of numbers. The standard mathematical techniques for interpolation were unsatisfactory for the problems presented by industry. Mathematically, they were sound but numerically they were unstable and prone to failure. New methods were soon discovered by many researchers working in R&D laboratories for companies. The principal methods were developed by two French automobile companies Citroën and Rénault by the mathematician Paul de Faget de Casteljau and the engineer Pierre Bézier. Other methods were developed by US and UK companies by Ferguson at Boeing and Coons at MIT, and A.R. Forest at Cambridge and M. Sabin at British Aircraft Corporation.

The modern theory of geometric modelling is still a young field (under 50 years as a formal discipline). New methods and applications arise every day, and the algorithms that are included in these notes are being refined and modified for each new problem and application.