Homework 1 Additional Problems

Recall that in a bar lying along the x-axis from $x = a$ to $x = b$ the temperature $u(x,t)$ (here t is time) obeys the PDE

$$u_t - \alpha u_{xx} = 0 \tag{1}$$

where α is the thermal diffusivity of the bar. For convenience let's assume that $\alpha = 1$.

To nail down a unique solution we also need to supply boundary conditions, for example, $u(a,t) = 0$ and $u(b,t) = 0$ (the temperature at both ends is kept at zero at all times). We also need to specify an initial condition, say $u(x,0) = u_0(x)$ for some given function $u_0(x)$.

1. Suppose the bar is defined by $0 \le x \le 1$ and $u_0(x) = \sin(\pi x)$. Verify that the function $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$ satisfies the heat equation (1) with the Dirichlet boundary conditions $u(0,t) = 0, u(1,t) = 0$, and of course the given initial condition. Plot the solution as a function of x on the interval $0 \le x \le 1$ at times $t = 0, 0.1$, and 0.5 (separate graphs for each). Interpret—what happens to the temperature in the bar?

2. Suppose we use "Neumann" or "insulating" boundary conditions $u_x(0,t) = 0$ and $u_x(1,t) = 0$ with initial data $u(x,0) = 2 + \cos(\pi x)$. Repeat the last exercise steps to show that $u(x,t) = 2 + e^{-\pi^2 t}\cos(\pi x)$ is the solution. Again, graph it at time $t = 0, 0.1, 0.5$. What is happening to the temperature of the bar?

3. Suppose the bar is unbounded, say spanning the whole x-axis from $-\infty < x < \infty$, so "boundary conditions" don't come into play (at least not obviously). Verify that the function

$$u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

satisfies the heat equation for $t > 0$ (take $\alpha = 1$ as above). Plot $u(x,t)$ as a function of x for $-5 \le x \le 5$ for $t = 0.001, 0.01, 0.1, 1.0$. Also, compute

$$\int_{-\infty}^{\infty} u(x,t)\,dx$$

for each t value (use Maple). What interpretation would you give to the "function" $u(x,0)$?