The Cross Product

Definition

The cross product is an operation on two three-dimensional vectors which results in a third vector orthogonal to the first two. The length of the cross-product is equivalent to the area of the parallelogram formed by the two vectors.

Here, the cross product \( \mathbf{u} \times \mathbf{v} \) would point out of the page, and would have magnitude equal to the area of the parallelogram shown.

The cross product of two vectors \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \) and \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) is defined as

\[
\mathbf{u} \times \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin \theta
\]

where \( \theta \) is the angle formed by the two vectors.

Determining Direction

The Right Hand Rule

To find the direction of \( \mathbf{u} \times \mathbf{v} \), begin with the fingers of your right hand pointing in the direction of \( \mathbf{u} \). Curl your fingers toward \( \mathbf{v} \). Your thumb is now pointing in the direction of the cross product. Notice that \( \mathbf{v} \times \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \) are not the same.

Determining Sign

By convention, we follow the arrows in the following diagram to determine the cross product of two basis vectors. Moving with the arrows gives a positive result, and against the arrows gives a negative result. Crossing two vectors that point in the same direction gives zero.

- \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), because it goes with the arrows.
- \( \mathbf{k} \times \mathbf{j} = -\mathbf{i} \), because we move against the arrows.
- \( \mathbf{j} \times \mathbf{j} = \mathbf{0} \), because they are in the same direction.
Calculating the Cross Product

The cross product of two vectors \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \) and \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) can be calculated by either of the following methods.

**Method 1: Diagonals**

First set up the following matrix:

\[
\begin{bmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3
\end{bmatrix}
\]

Rewrite the first two columns next to the matrix, and multiply along the following diagonals.

Sum the products to obtain \((u_2 v_3)i + (u_3 v_1)j + (u_1 v_2)k\). Then multiply again along the following diagonals:

Subtract these terms to obtain the cross product:

\[
\mathbf{u} \times \mathbf{v} = (u_2 v_3)i + (u_3 v_1)j + (u_1 v_2)k - (u_2 v_3)i - (v_2 u_3)i - (v_3 u_1)j - (v_1 u_2)k
\]

\[
= (u_2 v_3 - v_2 u_3)i + (u_3 v_1 - v_3 u_1)j + (u_1 v_2 - v_1 u_2)k
\]

**Method 2: Determinants**

First set up the following matrix:

\[
\begin{bmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3
\end{bmatrix}
\]
For each basis vector, \( i, j, \) and \( k \), cross out the row and column containing that vector, as shown:

\[
\begin{bmatrix}
1 & j & k \\
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3
\end{bmatrix}
\quad \begin{bmatrix}
i & j & k \\
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3
\end{bmatrix}
\quad \begin{bmatrix}
i & j & k \\
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3
\end{bmatrix}
\]

Find the determinant of each remaining \( 2 \times 2 \) matrix. Use them as coefficients in the following formula for the cross product. (Don’t forget the negative sign in front of \( j \).)

\[
\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \mathbf{k}
\]

\[
= (u_2 v_3 - v_2 u_3) \mathbf{i} - (v_3 u_1 - u_3 v_1) \mathbf{j} + (u_1 v_2 - v_1 u_2) \mathbf{k}
\]

Notice that both methods yield the same formula for the cross product.

**Practice Problems**

1. Use the right hand rule to determine the direction of each cross product.
   a. 
   b. 
   c. 

   \[
   \begin{array}{c}
   u \\
   \mathbf{v}
   \end{array}
   \quad \begin{array}{c}
   \mathbf{a} \\
   \mathbf{b}
   \end{array}
   \quad \begin{array}{c}
   x \\
   y
   \end{array}
   \]

   \[
   \mathbf{u} \times \mathbf{v} \quad \mathbf{b} \times \mathbf{a} \quad \mathbf{x} \times \mathbf{y}
   \]

2. Determine the following cross products using the correct sign convention:
   a. \( \mathbf{j} \times \mathbf{i} = \) 
   b. \( \mathbf{k} \times \mathbf{i} = \) 
   c. \( \mathbf{j} \times \mathbf{k} = \) 
   d. \( \mathbf{i} \times \mathbf{i} = \)

3. Use the “diagonals” method to find the cross product \( \mathbf{u} \times \mathbf{v} \) of the vectors:
   a. \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} \) \quad \( \mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \)
   b. \( \mathbf{u} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} \) \quad \( \mathbf{v} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \)
4. Use the “determinants” method to find the cross product $\mathbf{u} \times \mathbf{v}$ of the vectors:

a. $\mathbf{u} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ \hspace{1cm} $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

b. $\mathbf{u} = -\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ \hspace{1cm} $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Solutions to Practice Problems: