Basics of Differentiation – Review

The Limit Definition

\[
\frac{d}{dx} f(x) = \lim_{h \to 0} \left( f \left( \frac{f(x+h)-f(x)}{h} \right) \right)
\]

The limit definition is the most basic formula for calculating the derivative of a function. In this formula, \( f(x) \) is the function to be differentiated, and \( h \) is a small change in \( x \).

The Power Rule

In many cases, the limit definition can be bypassed in favor of the power rule. For any function \( a \cdot x^n \), the derivative can be found as:

\[
\frac{d}{dx} a \cdot x^n = a \cdot n \cdot x^{n-1}
\]

To apply this rule, multiply the coefficient, \( a \), by the current exponent, \( n \), and then decrease the exponent on \( x \) by 1.

Derivative of a Sum

The derivative of a sum of terms is equal to the sum of the derivatives of the terms.

\[
\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).
\]

This makes it easy to differentiate polynomials using the power rule.

Example: \( \frac{d}{dx} (2x^3 + x^7) = \frac{d}{dx} (2x^3) + \frac{d}{dx} (x^7) = 6x^2 + 7x^6 \)

The Product Rule

For two terms multiplied by one another, the power rule doesn’t apply. Instead, the product rule can be used:

\[
\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx} (g(x)) + \frac{d}{dx} (f(x)) \cdot g(x)
\]

To apply this rule, differentiate each function separately. Multiply the first function by the derivative of the second, and vice-versa. Then add the two terms together.
**The Quotient Rule**

For two terms divided by one another, the quotient rule is a convenient shortcut.

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}
\]

Again, differentiate each function separately. Then multiply the numerator function by the derivative of the denominator and vice versa. Subtract the two terms, as shown. Divide by the square of the denominator.

**The Chain Rule**

For a composition of two functions, \(f(x)\) and \(g(x)\), use the chain rule.

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)
\]

To use this formula, first find the derivative of the inside and outside functions separately. Apply the derivative of the outside function to the inside function. Then multiply by the derivative of the inside function.

**Example:** \((2x + 1)^3\)

Inside function: \(2x + 1\)  
\[\frac{d}{dx} (2x + 1) = 2\]

Outside function: \(y^3\)  
\[\frac{d}{dx} (y^3) = 3y^2\]

\[
\frac{d}{dx} ((2x + 1)^3) = 3(2x + 1)^2 \cdot 2 = 6(2x + 1)^2
\]

**Practice Problems:**

1.) Find, using the limit definition, the derivative of \(x + 2\).

2.) Find the derivative of:
   a. \(x^2\)
   b. \(2x^{-5}\)
   c. \(3x^3\)
   d. \(2x^4 - x^3 + 5x\)
   e. 5 (Think of this as \(5x^0\).)
3.) Use the product rule to find the derivative of:
   a. \((x+3)(x+4)\)
   b. \(x^3(x^2+3x)\)

4.) Use the quotient rule to find the derivative of:
   a. \(\frac{7x+3}{x+2}\)
   b. \(\frac{1}{x^2}\)

5.) Use the chain rule to find the derivative of:
   a. \((7x + 1)^3\)
   b. \(3(x^3 + 6)^2 + 2\)

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**Practice Problem Solutions:**

\[
\begin{align*}
\varepsilon^x \varepsilon \cdot (9 + \varepsilon x) & \quad q \\
\varepsilon (1 + x\varepsilon) & \quad \varepsilon \\
x & \quad q \\
\frac{\varepsilon (\varepsilon + x) - (\varepsilon + x)\varepsilon}{(\varepsilon + x)(\varepsilon - x)} & \quad \varepsilon \\
(\varepsilon + x\varepsilon)\varepsilon x + (\varepsilon x + \varepsilon x)\varepsilon x & \quad q \\
\varepsilon & \quad \varepsilon \\
0 & \quad \varepsilon \\
\varepsilon + \varepsilon x - \varepsilon x & \quad p \\
\varepsilon & \quad \varepsilon \\
\varepsilon \varepsilon & \quad \varepsilon \\
x & \quad \epsilon \\
1 & \quad \epsilon
\end{align*}
\]