Rose-Hulman Undergraduate Mathematics Journal

E-ergodicity and speedups

Tyler B. George[a]

Volume 16, No. 1, Spring 2015

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Ferris State University

Rose-Hulman Undergraduate Mathematics Journal
Volume 16, No. 1, Spring 2015

E-ergodicity and speedups

Tyler B. George

Abstract. We introduce the notion of "E-ergodicity" of a measure-preserving dynamical system (where E is a subset of $\mathbb{N}$). We show that given an E-ergodic system T and aperiodic system S, T can be sped up to obtain an isomorphic copy of S, using a function taking values only in E. We give examples applying this concept to the situation where E is a congruence class, the image of an integer polynomial, or the prime numbers.

Acknowledgements: I thank Dr. David McClendon for his constant support and expertise. Without his efforts, this research would not have been possible.

1 Introduction

Stock prices, weather patterns, population changes, and gas prices are all examples of sets of quantities that change as time passes, and thus can be modeled with dynamical systems. A *dynamical system*, simply speaking, consists of a set and a transformation on that set. The set, which we usually denote by X, is called the *phase space* and represents all possible states of the system at any particular time. The transformation $T : X \to X$ describes how one state evolves to another state. Thus if the current state $x \in X$ is a stock's price today, then $T(x)$ might give tomorrow's stock price, and the set X would be the set of all theoretically possible prices of the stock. One obvious goal given such a setup would be to predict the stock price in the distant future.

It seems at first that dynamical systems coming from different areas like economics, physics or biology might have nothing in common. However, in many cases, the dynamical systems modeling these natural phenomena have similarities which are only seen when we look at the system *abstractly*. For example: take a dynamical system (maybe our example from above that modeled stock prices) and forget that x represented the stock price (just think of x as a generic variable). What remains is an abstract dynamical system, modeled completely by an abstract mathematical function T. To a layman this might seem like madness, but stripping away the application of a system allows us to discover its intrinsic mathematical qualities, and to determine if two such systems are the "same" or "different".

There are many different notions of "sameness" in mathematics: our notion is *isomorphism*. This means formally that there is a certain kind of map (described later) from one system's phase space to the other that preserves the dynamics. Unfortunately, determining if two systems are isomorphic is difficult, because constructing such a map directly or showing that no such map can exist is hard. To attack this problem, we try to discover mathematical properties systems may or may not possess, called *invariants*. A property of a dynamical system is an invariant if whenever two systems are isomorphic, either they both have the property or neither of them have the property. We can tell that two systems are not isomorphic by finding an invariant which describes one system but not the other. Another advantage of thinking about invariants is that if one could show that a new, as-yet-unstudied system is mathematically the "same" as some system that has been thoroughly researched, then the invariants of the previously studied system would be known for the new system.

One useful invariant of a dynamical system is *ergodicity*. An ergodic dynamical system, loosely speaking, is one which has the following two equivalent properties:

- the system cannot be broken into two or more nontrivial subsystems which do not interact with one another;

- the "time averages" of any measurement on the the system converge to the "space average" of the measurement at any instant.

Ergodicity has been widely studied, and is the cornerstone of the branch of mathematics called ergodic theory. (For more on ergodic theory, consult the texts by Petersen [P], Silva [S], and Walters [W].) This area of math has connections with geometry, number theory, graph theory, and harmonic analysis, as well as physics, biology, economics and other fields. The key idea of this paper is a new invariant related to ergodicity, which we call "E-ergodicity". In Section 2 we'll introduce the necessary background information for the understanding of this paper and introduce our results. Then, in Section 3 we define our new concept *E-ergodicity* along with its relevance to our results, and give the necessary proof for our theorem in Section 4. Lastly, in Section 5 we end with some questions that still need solutions.

2 Background

Arnoux, Ornstein, and Weiss described in a 1985 paper [AOW] what we now call a "speedup" of a measure-preserving dynamical system. They showed that for an ergodic transformation $(X, \mathcal{X}, \mu, T)$ and an aperiodic measure-preserving transformation $(Y, \mathcal{Y}, \nu, S)$, there exists a measurable function (noting that in this paper we will not consider 0 a natural number) $p : X \to \mathbb{N}$ such that $(X, \mathcal{X}, \mu, T^{p(x)})$ is isomorphic to $(Y, \mathcal{Y}, \nu, S)$. In this 4-tuple, X is the collection of all possible "states" of a system, $\mathcal{X}$ is a σ-algebra, μ is a measure on $\mathcal{X}$ and $T : X \to X$ is a measure preserving transformation. Thus the function p "speeds up" T to behave like S. As such, all ergodic transformations comprise one equivalence class under this notion of "speedup equivalence".

In 2013, Babichev, Burton and Fieldsteel [BBF] gave a "relative" version of this result, proving that for any pair of aperiodic group extensions by a locally compact group G, if the first extension is ergodic, then it can be sped up to look like the second using a speedup function measurable with respect to the base factor. This work can be thought of as an extension Fieldsteel [F] and Gerber's [G] results on relative orbit equivalence, just as the Arnoux, Ornstein, and Weiss result can be thought of as an extension of Dye's Theorem [D1], [D2]. Recently, Johnson and McClendon [JM] extended both the Arnoux, Ornstein, Weiss machinery and the results of Babichev, Burton and Fieldsteel to measure-preserving actions of $\mathbb{Z}^d$, which itself is akin to the generalization of Dye's Theorem to actions of discrete amenable groups [CFW].

In this paper, we look further at speedups of single transformations and examine the conditions under which the range of the function p described above can be taken to be various proper subsets E of $\mathbb{N}$, rather than $\mathbb{N}$ itself. We define a notion called E-ergodicity and show that if T is E-ergodic, then T can be sped up to look like any aperiodic S using a speedup function taking values only in E. As applications of this idea, we show that if T^k is ergodic, then p can be chosen to take values in any congruence class modulo k; we show that if T is weak mixing then p can be chosen so that it takes values in the range of any integer polynomial; lastly, we show that if T is totally ergodic, then p can be chosen so that it takes values either in the range of any integer polynomial, or in any affine image of the prime numbers. These results are described in detail at the end of this section.

We begin with preliminary definitions. First, we specify our universe of discourse: we study the dynamics of functions which preserve a standard probability measure. More precisely:

Definition 2.1. *A* Lebesgue probability space *is a measure space isomorphic to the unit interval with the usual Lebesgue measure.*

Definition 2.2. *A* measure-preserving dynamical system $(X, \mathcal{X}, \mu, T)$ *is a Lebesgue probability space* $(X, \mathcal{X}, \mu)$ *together with a transformation* $T : X \to X$ *which is* $\mathcal{X}-$*measurable and preserves* μ *(that is* $\mu(T^{-1}(A)) = \mu(A)$ *for all* $A \in \mathcal{X}$*).*

We often refer to such systems as (X, T) or T, and in this paper, we assume that all transformations under consideration are invertible. Since we are interested in studying the dynamics of iteration by T, throughout this paper $T^k(x)$ represents $(T \circ T \circ \cdots \circ T)(x)$ where there are k Ts in the iteration.

An measure-preserving system T is called *aperiodic* if its set of periodic points has measure zero, that is

$$\mu\left(\bigcup_{n\in\mathbb{Z}-\{0\}} \{x \in X : T^n(x) = x\}\right) = 0.$$

An measure-preserving system T is called *ergodic* if for any $A \in \mathcal{X}$ satisfying $T^{-1}(A) = A$ almost surely, either $\mu(A) = 0$ or $\mu(A^C) = 0$. Notice that since we assume X is a Lebesgue probability space, if T is ergodic, then T is also aperiodic. Observe also that for any k in the natural numbers, if T^k is ergodic then T is ergodic.

Definition 2.3. *We say a measure-preserving system* $(X, \mathcal{X}, \mu, T)$ *is* totally ergodic *if for all* $k \in \mathbb{N}$*,* T^k *is ergodic.*

Definition 2.4. *We say two measure-preserving systems* $(X, \mathcal{X}, \mu, T)$ *and* $(Y, \mathcal{Y}, \nu, S)$ *are* isomorphic *(and write* $(X, T) \cong (Y, S)$ *or* $T \cong S$*) if*

1. *there exist measurable sets* $X_0 \subseteq X$ *and* $Y_0 \subseteq Y$ *with* $\mu(X_0) = \nu(Y_0) = 1$*,* $T(X_0) \subseteq X_0$ *and* $S(Y_0) \subseteq Y_0$*, and*

2. *there exists a measurable bijection* $\phi : X_0 \to Y_0$ *called an* isomorphism*,*

such that

(a) $\mu(\phi^{-1}(A)) = \nu(A)$ *for all measurable* $A \subseteq Y_0$*, and*

(b) $\phi(T(x)) = S(\phi(x))$ *for all* $x \in X_0$*.*

This paper is concerned with speedups of measure-preserving systems, defined as follows:

Definition 2.5. *Given an measure-preserving system* $(X, \mathcal{X}, \mu, T)$*, a* speedup *of* T *is another measure-preserving system* $(X, \mathcal{X}, \mu, \overline{T})$ *where* $\overline{T} = T^{p(x)}$ *for some measurable* $p : X \to \mathbb{N}$*. The function* p *is called the* speedup function *of* $\overline{T}$*.*

Notice that speedups must be $1-1$ almost surely and preserve μ.

Definition 2.6. *Given measure-preserving systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$, and given a subset E of $\mathbb{N}$, we say T can be E-*sped up *to S, and write $T \overset{E}{\rightsquigarrow} S$, if there exists a speedup $\overline{T}$ of T such that*

1. *$\overline{T} \cong S$; and*

2. *the speedup function p of $\overline{T}$ takes values only in E.*

Given this notation, following from Arnoux, Ornstein and Weiss's results [AOW], is that if T is ergodic and S is aperiodic, $T \overset{\mathbb{N}}{\rightsquigarrow} S$ (so in this setting $\overset{\mathbb{N}}{\rightsquigarrow}$ is an equivalence relation on the space of ergodic transformations). In this paper we look at situations when E is a proper subset of $\mathbb{N}$, and we prove:

Theorem 2.7. *Let $E \subseteq \mathbb{N}$. Also let $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ be measure-preserving systems where T is E-ergodic and S is aperiodic. Then $T \overset{E}{\rightsquigarrow} S$.*

This theorem applies in many settings. Here is a list of applications which we will prove:

Corollary 2.8. *Let $k \in \mathbb{N}$. Also let $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ be measure-preserving systems where T^k is ergodic. Then for any $a \in \{0, 1, 2, ...\}$, $T \overset{k\mathbb{N}+a}{\rightsquigarrow} S$.*

Definition 2.9. *An* integer polynomial *is a polynomial taking integer values on the integers.*

A simple example of an integer polynomial is $f(x) = x^2$; since for any integer value of x, $f(x)$ is also an integer. Indeed, any polynomial with integer coefficients is an integer polynomial. However, integer polynomials do not need to have integer coefficients. For example, the function $g(x) = \frac{x^2+x}{2}$, whose coefficients are almost all $\frac{1}{2}$, is an integer polynomial.

Corollary 2.10. *Let $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ be measure-preserving systems where T is totally ergodic. Then for any integer polynomial p, $T \overset{p(\mathbb{N})}{\rightsquigarrow} S$.*

A measure-preserving system T is called *weak mixing* if $(X \times X, T \times T)$ is ergodic. If a system T is weak mixing, then it is ergodic.

Corollary 2.11. *Let $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ be measure-preserving systems where T is weak mixing. Then for any integer polynomial p, $T \overset{p(\mathbb{N})}{\rightsquigarrow} S$.*

Corollary 2.12. *Let $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ be measure-preserving systems where T is totally ergodic. Then for the set of primes $\mathbb{P}$, $T \overset{k\mathbb{P}+a}{\rightsquigarrow} S$ for any $k \in \mathbb{N}$ and any $a \in \{0, 1, 2, ...\}$.*

Similar to Babichev, Burton and Fieldsteel [BBF], and Johnson and McClendon [JM] method's, to prove Theorem 2.7 we will obtain the desired speedup of T and the isomorphism from it to S as limits of sequences of partially defined speedups and isomorphisms.

In Section 3, we define E-ergodicity and show many examples, from which the corollaries above will follow once Theorem 2.7 is proven. In Section 4, we will prove Theorem 2.7.

3 E-ergodicity

The key idea of this paper is E-ergodicity, as defined here:

Definition 3.1. *Let $(X, \mathcal{X}, \mu, T)$ be an measure-preserving system and let $E \subseteq \mathbb{N}$. We say T is E-*ergodic *if for all $A, B \subseteq X$ of positive measure, there exists $A' \subseteq A$ and $i' \in E$ such that $\mu(A') > 0$ and $T^{i'}(A') \subseteq B$.*

Notice that T is ergodic if and only if it is $\mathbb{N}$-ergodic. For if T is ergodic, then for $A, B \subseteq X$ of positive measure, by the pointwise ergodic theorem,

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathbb{1}_B(T^n(x)) \to \int \mathbb{1}_B d\mu = \mu(B) > 0$$

almost surely. Then, for almost every $x \in A$, there is $i \in \mathbb{N}$ such that $T^i(x) \in B$. Since there are only countably many i, there exists an $i' \in \mathbb{N}$ such that

$$A' = \{x \in A : T^{i'}(x) \in B\}$$

has positive measure. Thus T is $\mathbb{N}$-ergodic. Conversely, if T is not ergodic, there is an invariant set D such that $0 < \mu(D) < 1$. Choose $A \subseteq D, B \subseteq X - D$ of positive measure; there will be neither $A' \subseteq A$ nor $i' \in \mathbb{N}$ such that $T^{i'}(A') \subseteq B$, so therefore T is not $\mathbb{N}$-ergodic.

The next three lemmas show how E-ergodicity is affected when the set E is translated and/or multiplied by a constant.

Lemma 3.2. *Suppose T is E-ergodic. Then T is $(E + a)$-ergodic for any $a \in \{0, 1, 2, ...\}$.*

Proof. By E-ergodicity applied to $T^a(A)$ and B, there exists $A'' \subseteq T^a(A)$, $i'' \in E$ such that $\mu(A'') > 0$ and $T^{i''}(A'') \subseteq B$. Now let $A' = T^{-a}(A'')$, and let $i' = i'' + a$. Then the result follows. □

Lemma 3.3. *Let $k \in \mathbb{N}$. Suppose T^k is E-ergodic. Then T is (kE)-ergodic.*

Proof. By E-ergodicity of T^k applied to A and B, there exists $A' \subseteq A$, $i'' \in E$ such that $\mu(A') > 0$ and $(T^k)^{i''}(A') \subseteq B$. Now let $i' = ki''$. Then the result follows. □

Lemma 3.4. *Let $a \in \{0, 1, 2, ...\}$ and $k \in \mathbb{N}$. Suppose T^k is E-ergodic. Then T is $(kE + a)$-ergodic. Consequently, if T^k is ergodic, then T is $(kE + a)$-ergodic.*

Proof. Combining Lemma 3.2 and Lemma 3.3, the result follows. □

We now give a series of results which characterize examples of E-ergodic systems. By applying results from the theory of multiple and other unconventional ergodic averages, we can show that classes of systems are E-ergodic for various sets E. First, we deal with the range of an integer polynomial:

Lemma 3.5. *Let $(X, \mathcal{X}, \mu, T)$ be such that T is totally ergodic, and let p be an integer polynomial. Then T is $p(\mathbb{N})$-ergodic.*

Proof. Frantzikinakis and Kra's [FK] Theorem 1.1 (using a single function $\mathbb{1}_B$, and single polynomial p), gives us the following:

$$\frac{1}{N}\sum_{i=0}^{N-1} \mathbb{1}_B(T^{p(i)}(x)) \xrightarrow{L^2} \int \mathbb{1}_B d\mu.$$

Then there exists a subsequence $\{N_k\}$ such that for $\mu-$almost every x,

$$\frac{1}{N_k}\sum_{i=0}^{N_k-1} \mathbb{1}_B(T^{p(i)}(x)) \to \int \mathbb{1}_B d\mu = \mu(B) > 0.$$

Therefore, we see that for a.e. $x \in X$, there exists i such that $\mathbb{1}_B(T^{p(i)}(x)) > 0$. Now, for all $i \in \mathbb{N}$, let $A_i = \{x \in A : T^{p(i)}(x) \in B\}$. Notice $\bigcup_i A_i = A$ almost surely, so there exists $j \in \mathbb{N}$ such that $\mu(A_j) > 0$. Set $A' = A_j$ and $i' = p(j)$. We have $A' \subseteq A$, and $T^{i'}(A') \subseteq B$ as desired. $\square$

To highlight why we require total ergodicity in Lemma 3.5, consider the following example: $p(x) = x^2 + x$. Then $p(n)$ is even for all $n \in \mathbb{N}$. Now, suppose there exists a T^2-invariant set A such that $0 < \mu(A) < 1$. Then A is invariant under any speedup of T^2, so no speedup of T^2 is ergodic. In this example, if S is ergodic, then S is not isomorphic to any speedup of T^2, so T cannot be $p(\mathbb{N})$-sped up to look like S. This same issue occurs whenever $p(\mathbb{N})$ is contained within any congruence class.

Lemma 3.6. *Let $(X, \mathcal{X}, \mu, T)$ be such that T is weak mixing, and let p be an integer polynomial. Then T is $p(\mathbb{N})$-ergodic.*

Proof. Bourgain's [B] Theorem 1 applied to $\mathbb{1}_B$, along with the logic used in Lemma 3.5, gives us our desired result. $\square$

Next, we deal with the prime numbers:

Lemma 3.7. *Let $(X, \mathcal{X}, \mu, T)$ be such that T is totally ergodic. Then for all $k \in \mathbb{N}$ and all $a \in \{0, 1, 2, ...\}$, T is $(k\mathbb{P} + a)$-ergodic, where $\mathbb{P}$ is the set of prime numbers.*

Proof. First, since T is totally ergodic, T^k is also totally ergodic. Thus Theorem 5 by Frantzikinakis, Host and Kra [FHK], (applied to functions $\mathbb{1}_B$ and 1) gives:

$$\frac{1}{\pi(N)}\sum_{\substack{i=0\\ i\in\mathbb{P}}}^{N-1} \mathbb{1}_B(T^{ki}(x)) \xrightarrow{L^2} \int \mathbb{1}_B d\mu = \mu(B) > 0.$$

where $\pi(N)$ is the number of primes less than or equal to N. Now, by the same logic as in the proof of Lemma 3.5, we can conclude T^k is $\mathbb{P}$-ergodic. Since T^k is $\mathbb{P}$-ergodic, T is $(k\mathbb{P} + a)$-ergodic for all $k \in \mathbb{N}$ and $a \in \{0, 1, 2, ...\}$ by Lemma 3.4. $\square$

4 Constructing the speedup

In this section, we prove Theorem 2.7, which says that for an E-ergodic T, T can be E-sped up to obtain an isomorphic copy of S. We begin with the following proposition, which shows that when T is E-ergodic, a partial speedup of T (whose speedup function takes values in E) can be found taking any subset of X to any other subset of equal size:

Proposition 4.1. *Let $E \subseteq \mathbb{N}$ and let $(X, \mathcal{X}, \mu, T)$ be such that T is E-ergodic. Also let $A, B \subseteq X$ be of equal positive measure. Then there is a measurable function*

$$p : A \to E$$

such that T^p is an isomorphism from A to B.

Proof. First, given subsets A and B of positive measure, we will say that a subset A' of A (or a pair (A', i') where $i' \in E$) is "E-good" if $T^{i'}(A) \subseteq B$. If T is E-ergodic, then E-good subsets always exist by definition.

Let

$$a_1 = \sup\{\mu(A') : A' \subseteq A \text{ and } A' \text{ is } E\text{-good}\}.$$

Now choose $A_1 \subseteq A$ and i_1 such that (A_1, i_1) is E-good and $\mu(A_1) > a_1 - 2^{-1}$. If $\mu(A_1) = \mu(A)$, we are done. If $\mu(A_1) < \mu(A)$, let

$$a_2 = \sup\{\mu(A') : A' \subseteq A - A_1 \text{ and } A' \text{ is } E\text{-good}\}$$

and choose $A_2 \subseteq (A - A_1)$ and i_2 such that (A_2, i_2) is E-good and $\mu(A_2) > a_2 - 2^{-2}$. If we continue this process, a pairwise disjoint sequence of sets $\{A_j\}$ and a sequence $\{i_j\} \subseteq E$ results.

Suppose that for all r, $\mu(\bigcup_{j=1}^{\infty} A_j) < \mu(A)$. In this case, by the definition of E-ergodicity, there is a set $A' \subseteq A - (\bigcup_{j=0}^{\infty} A_j)$ of positive measure and $i' \in \mathbb{N}$ such that (A', i') is E-good. But $\sum_{j=1}^{\infty} \mu(A_j) < \infty$, so $\mu(A_j) \to 0$, so $\mu(A_j) + 2^{-j} \to 0$, so for some j, we have

$$a_j < \mu(A_j) + 2^{-j} < \mu(A').$$

which contradicts the choice of a_j. Hence $\mu(\bigcup_{j=1}^{\infty} A_j) = \mu(A)$.

Define p by setting $p(x) = i_j$ whenever $x \in A_j$. This p satisfies the conclusions of the lemma. □

We will use the following terminology and notation in what follows, inherited from Arnoux, Ornstein and Weiss [AOW], Babichev, Burton and Fieldsteel[BBF], and Johnson and McClendon [JM]: a *Rokhlin tower* $\mathcal{T}$ for a measure-preserving system $(Y, \mathcal{Y}, \nu, S)$ is a pairwise disjoint collection $\{A_i\}_{i=1}^{h}$ of measurable subsets of Y such that for each i, $S(A_i) = A_{i+1}$. Each $A_i \in \mathcal{T}$ is called a *level* of $\mathcal{T}$, A_1 is the *base*, $h = h(\mathcal{T})$ is the *height*, and the common value $\nu(A_i)$ is the *width* of $\mathcal{T}$. We let $|\mathcal{T}| = \bigcup_{i=1}^{h} S^i(A_1)$ and $|\mathcal{T}|^0 = \bigcup_{i=1}^{h-1} S^i(A_1)$. A *column* of $\mathcal{T}$ is a tower of the form $\{S^i(B)\}_{i=0}^{h-1}$, where B is a measurable subset of the base of $\mathcal{T}$.

A *castle* for S is a finite collection $\mathcal{C} = \{\mathcal{T}_j\}_{j=1}^{J}$ of towers for S such that $|\mathcal{T}_{j_1}| \cap |\mathcal{T}_{j_2}| = \emptyset$ for all $j_1 \neq j_2$. We let $|\mathcal{C}| = \bigcup_{j=1}^{J} |\mathcal{T}_j|$ and $|\mathcal{C}|^0 = \bigcup_{j=1}^{J} |\mathcal{T}_j|^0$. We refer to $Y - |\mathcal{C}|$ as the *error set* of $\mathcal{C}$. A *level* of $\mathcal{C}$ (respectively a *column* of $\mathcal{C}$) is a level (resp. column) of a tower in $\mathcal{C}$. We denote set of all levels of $\mathcal{C}$ by $L(\mathcal{C})$.

If $\mathcal{T}$ is a tower for S, then each finite measurable partition $\mathcal{Q} = \{B_j\}_j$ of the base of $\mathcal{T}$ gives rise to a castle $\mathcal{T}_\mathcal{Q}$ whose towers are the columns of $\mathcal{T}$ with bases B_j. Given a finite partition $\mathcal{P}$ of $|\mathcal{T}|$, we obtain a partition $\mathcal{P}_\mathcal{T}$ of the base B of $\mathcal{T}$ whose atoms are maximal sets $\{B_j\}$ such that for every $i \in \{1, 2, ..., h-1\}, S^i(B_j)$ is contained in a single atom of $\mathcal{P}$. This partition yields a castle $(\mathcal{T})_{\mathcal{P}_\mathcal{T}}$ as above. We refer to this castle as the *castle of $\mathcal{P}$-columns* of $\mathcal{T}$. We make similar definitions for castles $\mathcal{C}$ and partitions of Y into the levels of $\mathcal{C}$ and the error set of $\mathcal{C}$.

Given two castles $\mathcal{C}_1$ and $\mathcal{C}_2$ for measure-preserving system S, we say $\mathcal{C}_2$ is *obtained from $\mathcal{C}_1$ by cutting and stacking* if

1. $|\mathcal{C}_1|^0 \subseteq |\mathcal{C}_2|^0$;
2. there is a finite partition $\mathcal{Q}$ of the bases of the towers of $\mathcal{C}_1$ such that each level of the castle $(\mathcal{C}_1)_\mathcal{Q}$ is a level of $\mathcal{C}_2$; and
3. for each tower of $(\mathcal{C}_1)_\mathcal{Q}$, there is a tower of $\mathcal{C}_2$ that contains it.

Note that condition (1) implies that if $\{A_i\}_{i=1}^{h_2}$ is a tower in $\mathcal{C}_2$ and A_j is a base of a tower of $(\mathcal{C}_1)_Q$ of height h_1, then we must have $j \leq h_2 - h_1$.

To explain some of this language, lets look at an example, the *dyadic odometer* described by Silva [S]. This is a transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(x) = \begin{cases} x + \frac{1}{2}, & \text{if } 0 \leq x < \frac{1}{2}; \\ x - \frac{1}{4}, & \text{if } \frac{1}{2} \leq x < \frac{3}{4}; \\ x - \frac{5}{8}, & \text{if } \frac{3}{4} \leq x < \frac{7}{8}; \\ \vdots & \end{cases}$$

The graph of T is shown in Figure 1. Observe x values in $[0, \frac{1}{2})$ are mapped by T to $[\frac{1}{2}, 1)$; x values in $[\frac{1}{2}, \frac{3}{4})$ are mapped to $[0, \frac{1}{4})$, etc. We now define this same transformation using cutting and stacking as defined above.

Figures 2 - 4 are a series of pictures that describe how this works. Start with an interval $[0, 1]$, which we consider to be a tower of height 1. Call the castle consisting of this single tower $\mathcal{C}_1$. To obtain $\mathcal{C}_2$ from $\mathcal{C}_1$, cut the interval from Figure 2 in half, and place the right half on top of the left half. Think of T as mapping each point directly upwards. So far, T is only defined on $[0, 1/2)$. To obtain $\mathcal{C}_3$ from $\mathcal{C}_2$, cut the entire stack from Figure 3 in half, and again place the right half on top of the left half. Think of T again as mapping each point directly upwards; now T is defined on $[0, 3/4)$. Repeat this process over and over; in the limit

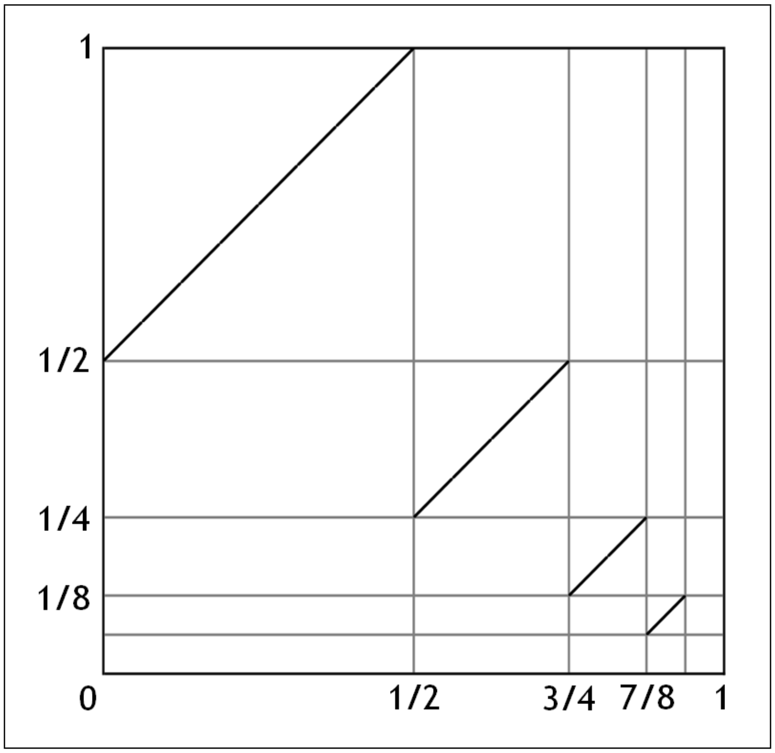

Figure 1: dyadic odometer

Figure 2: $\mathcal{C}_1$ for the dyadic odometer

we obtain the dyadic odometer which is eventually defined on $[0, 1)$. This is an aperiodic, measure-preserving transformation, and it turns out that every aperiodic dynamical system can be obtained by a general type of this construction.

Lemma 4.2 (Rohklin Tower Lemma [AOW])**.** *Let $(Y, \mathcal{Y}, \nu, S)$ be an aperiodic, measure-preserving dynamical system. Then there is a sequence $\{\mathcal{C}_l\}_{l=1}^{\infty}$ of castles, such that:*

1. *for each l, all towers in the castle $\mathcal{C}_l$ have the same height;*

2. *for each l, $\mathcal{C}_{l+1}$ is obtained from $\mathcal{C}_l$ by cutting and stacking;*

3. $\nu\left(\bigcup_{l=1}^{\infty} \mathcal{C}_l\right) = 1$*; and*

4. $\bigcup_{l=1}^{\infty} L(\mathcal{C}_l) = \mathcal{Y}$.

Last, we prove Theorem 2.7, which yields Corollaries 2.8, 2.10, 2.11 and 2.12 when combined with Lemmas 3.4, 3.5, 3.6 and 3.7, respectively. This proof mirrors Babichev, Burton

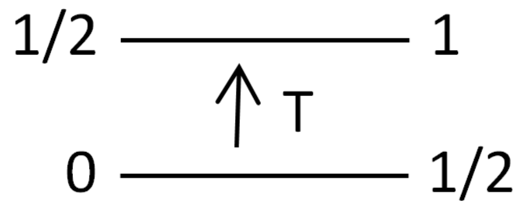

Figure 3: $\mathcal{C}_2$ for the dyadic odometer

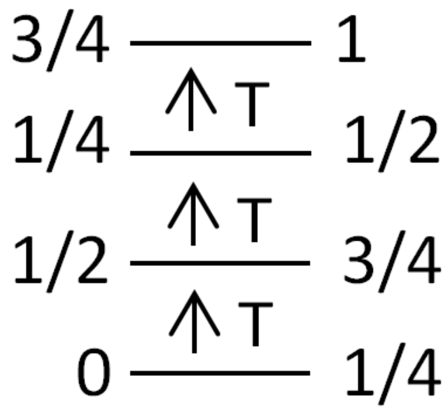

Figure 4: $\mathcal{C}_3$ for the dyadic odometer

and Fieldsteel's [BBF] Theorem 1 proof.

Proof of Theorem 2.7. We will obtain the desired relative speedup of T and the isomorphism from it to S as limits of sequences of partially defined speedups and isomorphisms. Start by choosing a sequence of castles $\{\mathcal{C}_l^S\}_{l=1}^{\infty}$ for S as in Lemma 4.2. Denote the towers of these castles by $\mathcal{T}_{l,j}^S$, and the levels of these towers by $A_{l,j,i}^S$. Thus $\mathcal{C}_l^S = \{\mathcal{T}_{l,j}^S\}_j$ and $\mathcal{T}_{l,j}^S = \{A_{l,j,i}^S\}_i$. Readers may find it helpful to follow along with Figure 5

The first step of the argument is to find a measurable function p_1, taking values in E, such that T^{p_1} is defined on a subset of X in a way that matches the action of S on the the levels of the first castle $\mathcal{C}_1^S$. Start by making a copy $\mathcal{C}_1^T$ of $\mathcal{C}_1^S$ in X. That is, choose arbitrary pairwise disjoint sets $A_{1,j,i}^T \in \mathcal{X}$ corresponding to the levels of $\mathcal{C}_1^S$ such that for each j and i, $\mu(A_{1,j,i}^T) = \nu(A_{1,j,i}^S)$. Fix j and an arbitrary isomorphism $\phi_{1,j} : A_{1,j,1}^T \to A_{1,j,1}^S$.

Apply Proposition 4.1 to each $A_{1,j,i}$ with $i \in \{1, ..., h_1 - 1\}$ to obtain a measurable function

$$p_{1,j,i} : A_{1,j,i} \to E$$

such that

$$T^{p_{1,j,i}} : A_{1,j,i} \to A_{1,j,i+1}$$

isomorphically. (The application of Proposition 4.1 is where we use the E-ergodicity of T.) Then, by letting $p_{1,j} = \bigcup_{i=1}^{h_1-1} p_{1,j,i}$ we obtain a partially defined speedup $T_1 = T^{p_{1,j}}$ of T, defined on $|\mathcal{T}_{1,j}|^0$, for which $\mathcal{T}_{1,j}$ is a tower.

Extend $\phi_{1,j}$ to $|\mathcal{T}_{1,j}|$ so that on all levels other than the top of $\mathcal{T}_{1,j}$,

$$\phi_{1,j} \circ T_1(x) = S \circ \phi_{1,j}(x)$$

almost surely. In particular, for each i, $\phi_{1,j}(A^T_{1,j,i}) = A^S_{1,j,i}$. Repeating this construction on each tower of $\mathcal{C}^T_1$, we set $\phi_1 = \bigcup_j \phi_{1,j}$ to obtain an isomorphism from $|\mathcal{C}^T_1|$ to $|\mathcal{C}^S_1|$ intertwining T_1 and S. See Figure 5

Next, we define a function p_2, extending p_1, so that T^{p_2} is defined on a larger subset of X. Fix an increasing sequence of finite partitions $\{\mathcal{P}_l\}_{l=1}^{\infty}$ of X that generate $\mathcal{X}$. Choose n_2 so that the partition $\phi_1(\mathcal{P}_1)$ is approximated to within $\frac{1}{2}$ (in the partition metric) by the levels of $\mathcal{C}^S_{n_2}$; for notational convenience re-index $\mathcal{C}^S_{n_2}$ as $\mathcal{C}^S_2$. Let $\mathcal{C}^T_2$ denote a copy of $\mathcal{C}^S_2$ which is the image of $\mathcal{C}^S_2$ under ϕ_1^{-1}, that is for each level $A^S_{2,j,i}$ of $\mathcal{C}^S_2$ contained in $|\mathcal{C}^S_1|$, the corresponding level $A^T_{2,j,i}$ of $\mathcal{C}^T_2$ is given by $A^T_{2,j,i} = \phi_1^{-1}(A^S_{2,j,i})$, and we choose arbitrary disjoint subsets of X (each disjoint from $|\mathcal{C}^T_1|$) of the appropriate measure to serve as $A^T_{2,j,i}$ when $A^S_{2,j,i}$ is not contained in $|\mathcal{C}^S_1|$.

Our goal is to extend T_1 to a transformation T_2 on $|\mathcal{C}^T_2|^0$, so that T_2 is again a partially defined speedup of T. Fix a tower $\mathcal{T}_{2,j}$ in $\mathcal{C}^T_2$ and suppose that $|\mathcal{T}_{2,j}| \cap |\mathcal{C}^T_1| \neq \emptyset$. Let $\phi_2 : A^T_{2,j,1} \to A^S_{2,j,1}$ be an arbitrary isomorphism.

For each level $A^T_{2,j,i}$ with $i < h_2$ such that at least one of the two sets $A^T_{2,j,i}$ and $A^T_{2,j,i+1}$ is disjoint from $|\mathcal{C}^T_1|$, apply Proposition 4.1 to obtain a measurable function

$$p_{2,j,i} : A^T_{2,j,i} \to E$$

such that

$$T^{p_{2,j,i}} : A^T_{2,j,i} \to A^T_{2,j,i+1}$$

isomorphically. For any level $A^T_{2,j,i}$ with $i < h_2$ such that both $A^T_{2,j,i}$ and $A^T_{2,j,i+1}$ come from $|\mathcal{C}^T_1|$, set $p_{2,j,i} = p_1$.

Perform this construction on each tower of $\mathcal{C}^T_2$ which meets $|\mathcal{C}^T_1|$. If $\mathcal{T}_{2,j}$ is a tower of $\mathcal{C}^T_2$ that does not meet $|\mathcal{C}^T_1|$, then employ the simpler construction that was used in the first stage of the proof to define T_2 on that tower. Therefore, the transformation $T_2(x) = T^{p_2}(x)$ (where $p_2 = \bigcup_{j,i} p_{2,j,i}$) is a partially defined speedup of T that agrees with T_1 on its domain.

This procedure can be repeated indefinitely to produce a sequence $\{\mathcal{C}^T_l\}$ of castles $\mathcal{C}^T_l$ in X for partially defined transformations T_l, where the levels of $\mathcal{C}^T_l$ approximate the partition $\mathcal{P}_{l-1}$ to within $\frac{1}{l}$, so that each T_l is a speedup of T defined on $|\mathcal{C}^T_l|^0$, each T_{l+1} extends T_l, and the transformation $\overline{T} = \bigcup_l T_l$ is therefore a speedup of T defined almost everywhere.

This construction also produces a sequence of isomorphisms $\phi_l : |\mathcal{C}^T_l| \to |\mathcal{C}^S_l|$ that intertwine T_l and S. In the construction of the ϕ_l we observe that each ϕ_{l+1} agrees set-wise with ϕ_l on the levels of $\mathcal{C}^T_l$. Since the σ-algebras $\mathcal{X}_l$ generated by the levels of $\mathcal{C}^T_l$ increase to $\mathcal{X}$, the maps ϕ_l determine an isomorphism $\phi = \lim_{l\to\infty} \phi_l$ between $\overline{T}$ and S. This completes the proof of Theorem 2.7. $\square$

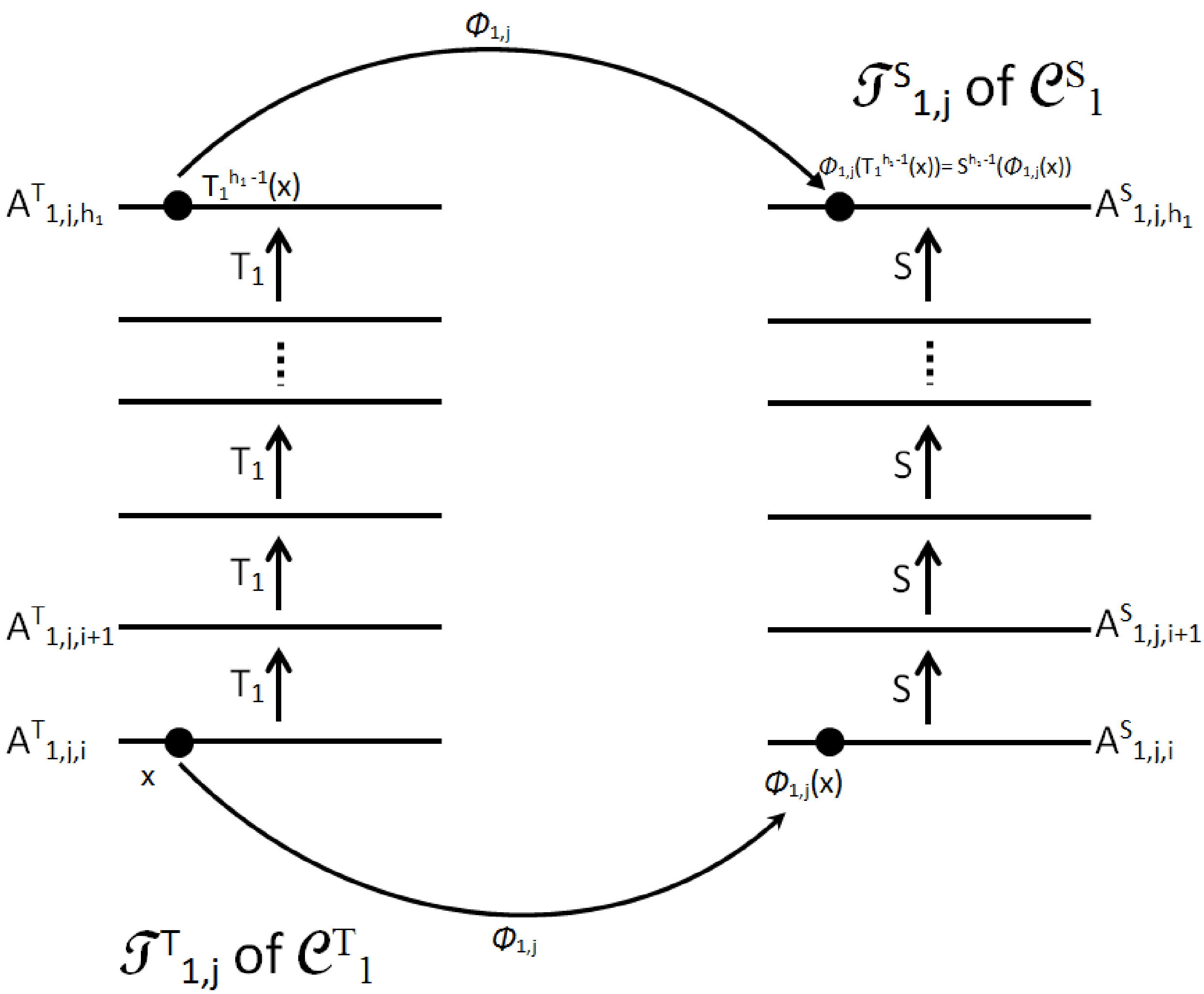

Figure 5: Proof 1

5 Open Questions

Here are some unanswered questions arising from our work. A few of these questions may be answerable by an undergraduate college student:

1. In Lemmas 3.5 and 3.7 of this paper, one could ask if total ergodicity is necessary. For example:
    - Is there an integer polynomial p for which every $p(\mathbb{N})$-ergodic system is totally ergodic?
    - Does there exist a $\mathbb{P}$-ergodic system which is not totally ergodic?
    - Given two "related" integer polynomials p and q, what is the relationship between $p(\mathbb{N})$-ergodic and $q(\mathbb{N})$-ergodic transformations?
2. There are many different conditions on a dynamical system equivalent to ergodicity. Can one define a similar set of equivalent conditions for E-ergodicity; in particular, is there such a version of the Birkhoff ergodic theorem (that is a generalization of arbitrary subsets of $\mathbb{N}$ defined by Bourgain [B]) connected to E-ergodicity?
3. Is there a reasonable definition of E-weak mixing or E-mixing?
4. Can one define notions of E-ergodicity for actions of d commuting measure-preserving transformations? Do the analogous results to those proven in this paper hold in that setting?
5. Do relative versions of this work hold for group extensions and/or finite extensions? (Almost assuredly the answer is yes by mimicking Babichev, Burton and Fieldsteel's [BBF] constructions).

References

[AOW] P. Arnoux, D. Ornstein and B. Weiss. Cutting and stacking, interval exchanges and geometric models. *Isr. J. Math.* **50** (1985), 160-168.

[B] J. Bourgain. Pointwise ergodic theorems for arithmetic sets. *Publ. Math. Inst. Hautes Études Sci.* **69** (1989), 5-45

[BBF] A. Babichev, R. Burton and A. Fieldsteel. Speedups of ergodic group extensions. *Ergodic Theory Dyn. Syst.* **33** (2013), 969-982.

[CFW] A. Connes, J. Feldman and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dyn. Syst.* **1** (1981), 431-450.

[D1] H.A. Dye. On groups of measure preserving transformations I. *Amer. J. Math.* **81** (1959), 119-159.

[D2] H.A. Dye. On groups of measure preserving transformations II. *Amer. J. Math.* **85** (1963), 551-576.

[F] A. Fieldsteel. Factor orbit equivalence of compact group extensions. *Isr. J. Math.* **38** (1981), 289-303.

[FHK] N. Frantzikinakis, B. Host and B. Kra. Multiple recurence and convergence for sequences related to the prime numbers. *J. Reine Angew. Math.* **611** (2007), 131-144.

[FK] N. Frantzikinakis and B. Kra. Polynomial averages converge to the product of integrals. *Isr. J. Math* **148** (2005), 267-276.

[G] M. Gerber. Factor orbit equivalence of compact group extensions and classification of finite extensions of ergodic automorphisms. *Isr. J. Math.* **57** (1987), 28-48.

[JM] A.S.A. Johnson and D. McClendon. Speedups of ergodic group extensions of $\mathbb{Z}^d-$actions. *Dyn. Sys.*, **29** (2014), 255-284.

[P] Petersen, K. *Ergodic Theory*, Cambridge: Cambridge University Press (1990).

[S] Silva, C. E. *Invitation to Ergodic Theory (Student Mathematical Library)*, Providence, RI: American Mathematical Society (2007).

[W] Walters, P. *An Introduction to Ergodic Theory (Grad. Texts in Math.)*, Springer (2000).