A short proof of Greenberg's Theorem

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March 16, 2020

Theorem (Greenberg, 1960)

Every countable group A is isomorphic to the automorphism group of a non-compact Riemann surface, with a finitely generated fundamental group if A is finite.

Theorem (Greenberg, 1973)

Every finite group is isomorphic to the automorphism group of a compact Riemann surface.

The proofs are long, complicated and non-constructive. In response to a question by Sasha Mednykh, I shall give a short and more constructive proof, using well-known properties of triangle groups, of part of Greenberg's results:

Theorem

Every finitely generated group A is isomorphic to the automorphism group of a Riemann surface, compact if A is finite.

Question Can one extend this proof to *all* countable groups A?

Surface kernel epimorphisms of triangle groups

Given any hyperbolic triangle group

$$\Delta := \Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle$$

(one with $l^{-1} + m^{-1} + n^{-1} < 1$), Dirichlet's Theorem on primes in arithmetic progressions implies that there exist surface kernel epimorphisms

$$\Delta \to G := PSL_2(q)$$

for infinitely many prime powers q such that

$$l, m, n \mid \frac{q+1}{2}.$$

The subgroups fixing ∞

$$\begin{array}{ccc} \Delta(l,m,n) & \longrightarrow & G = PSL_2(q) \\ \Big| \, q+1 & & \Big| \, q+1 \\ \Pi_g \cong N & \longrightarrow & G_\infty \end{array}$$

G acts primitively on the projective line $\mathbb{P}^1(q) = \mathbb{F}_q \cup \{\infty\}$.

The stabiliser G_∞ of ∞, and hence its inverse image N in Δ, are maximal subgroups of index $q+1$ in G and Δ.

Since $l, m, n \mid (q+1)/2$, the elliptic generators X, Y, Z of Δ act freely on $\mathbb{P}^1(q)$, so N is a surface group Π_g, of genus

$$g = \frac{q+1}{2}\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) + 1$$

by the Riemann–Hurwitz formula.

The epimorphism θ

$$\begin{array}{ccccc} & & \Delta(l,m,n) & \longrightarrow & G = PSL_2(q) \\ & & \Big| \, q+1 & & \Big| \, q+1 \\ A & \xleftarrow{\theta} & \Pi_g \cong N & \longrightarrow & G_\infty \end{array}$$

Given any finitely generated group A, choose q large enough that

$$g \geq d := \mathrm{rank}(A).$$

Since N is a surface group

$$N = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_i [A_i, B_i] = 1 \rangle$$

there is an epimorphism $\theta : N \to A$ with $A_i \mapsto$ generators, $B_i \mapsto 1$.

The kernel of θ

$$
\begin{array}{ccccc}
 & & \Delta(l,m,n) & \longrightarrow & G = PSL_2(q) \\
 & & \Big| \, q+1 & & \Big| \, q+1 \\
A & \overset{\theta}{\longleftarrow} & \Pi_g \cong N & \longrightarrow & G_\infty \\
\Big| & & \Big| & & \\
1 & \longleftarrow & M & &
\end{array}
$$

Define $M := \ker\theta$, so $M \triangleleft N$ with $N/M \cong A$.

Then $N \le N_\Delta(M) \le \Delta$, and the maximality of N in Δ implies that $N_\Delta(M) = N$ (we can choose θ so that M is not normal in Δ).

$A \cong \mathrm{Aut}\,\mathcal{H}$

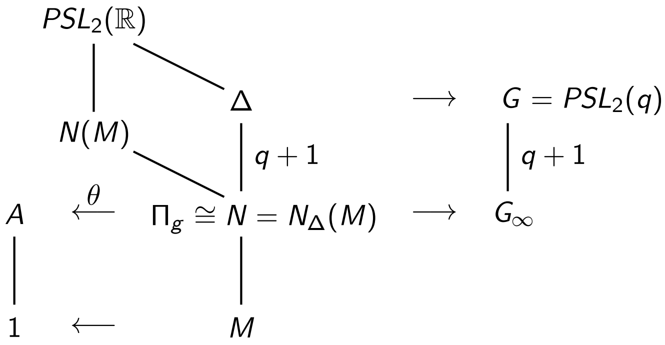
$$
\begin{array}{ccccccc}
 & & \Delta(l,m,n) & \longrightarrow & G = PSL_2(q) \\
 & & \Big| \; q+1 & & \Big| \; q+1 \\
A & \overset{\theta}{\longleftarrow} & \Pi_g \cong N = N_\Delta(M) & \longrightarrow & G_\infty \\
\Big| & & \Big| & & \\
1 & \longleftarrow & M & &
\end{array}
$$

Since

$$A \cong N/M = N_\Delta(M)/M,$$

we have realised A as $\mathrm{Aut}\,\mathcal{H}$ where $\mathcal{H}$ is the oriented hypermap (bipartite map) of type (l, m, n) corresponding to the subgroup M.

If A is finite, so is $|\Delta : M|$, so $\mathcal{H}$ is compact, i.e. a dessin d'enfant.



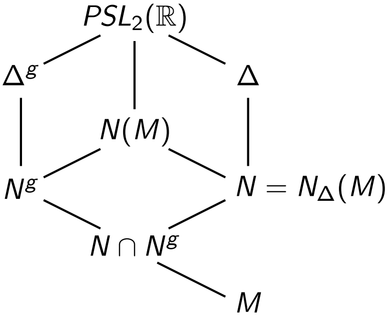
However, to prove Greenberg's Theorem we want to show that

$$A \cong \mathrm{Aut}\,\mathcal{S} \cong N(M)/M,$$

where $\mathcal{S}$ is the Riemann surface $\mathbb{H}/M$ and $N(M) := N_{PSL_2(\mathbb{R})}(M)$ (note that M is torsion-free!), so we need to show that

$$N(M) = N.$$

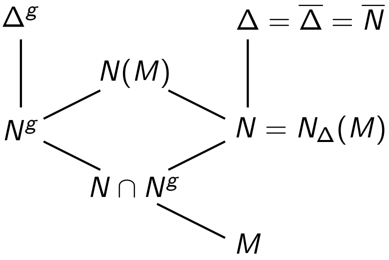
Proving that $N(M) = N$



Let $g \in N(M)$. Then $M \triangleleft N$ implies $M = M^g \triangleleft N^g$, so

$$N, N^g \leq N(M).$$

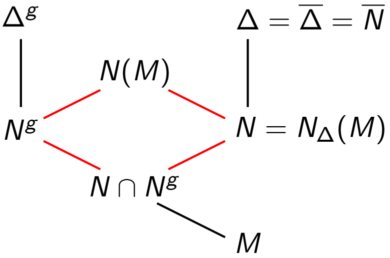
Commensurators



For most (l, m, n), e.g. $(2, 3, 13)$, Δ is maximal (Singerman 1972) and non-arithmetic (Takeuchi 1977). Then the commensurator

$$\overline{\Delta} := \{c \in PSL_2(\mathbb{R}) \mid |\Delta : \Delta \cap \Delta^c|, |\Delta^c : \Delta \cap \Delta^c| < \infty\}$$

is a Fuchsian group (by non-arithmeticity, Margulis 1991) containing Δ. Hence $\overline{\Delta} = \Delta$ (by maximality) and so $\overline{N} = \Delta$ also.



N, N^g and $N(M)$ are cocompact, so $|N(M) : N|$ and $|N(M) : N^g|$ are finite, and hence so are $|N : N \cap N^g|$ and $|N^g : N \cap N^g|$.

Thus $g \in \overline{N} = \Delta$, so we have proved that $N(M) \leq \Delta$. Hence

$$N(M) = N_\Delta(M) = N,$$

giving the required isomorphism

$$\mathrm{Aut}\, \mathcal{S} \cong N(M)/M = N/M \cong A.$$

Extension to countable groups of infinite rank?

$$
\begin{array}{ccc}
 & & \Delta(l,m,n) \\
 & & \Big| \; \infty \\
A & \stackrel{\theta}{\longleftarrow} & N = N_\Delta(M) \\
\Big| & & \Big| \\
1 & \longleftarrow & M
\end{array}
$$

If A has infinite rank, so must N, so N must be maximal of infinite index in Δ. There are plenty of these (J., J. Group Theory, 2019) with epimorphisms onto F_∞ and hence onto A. Thus

$$A \cong N/M = N_\Delta(M)/M \cong \mathrm{Aut}\,\mathcal{H}.$$

However, taking N of infinite index loses commensurability properties, so it's not clear how to ensure that $A \cong \mathrm{Aut}\,\mathcal{S}$.

THANK YOU FOR LISTENING!