On Large Groups of Automorphisms of Riemann Surfaces

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In Memoriam of Bill Harvey

Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Group and Related Topics

AMS Spring Eastern Sectional Meeting
Medford MA, March 2020

Given an orientable, closed surface X of genus $g \geq 2$ The equivalence:

$(X, \mathcal{M}(X)$, complex atlas) ($\mathcal{M}(X) = \langle x, y \rangle$, $p(x, y) = 0$, the field of meromorphic functions on X)

$X \equiv \frac{\mathbb{H}}{\Delta}$, with Δ a (cocompact) Fuchsian group
Δ discrete subgroup of $PSL(2, \mathbb{R})$

$(X, \mathcal{M}(X)$, complex curve) ($\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y)$, the field of rational functions on X)
The curve X given by the polynomial $p(x, y)$ and the meromorphic function $x : X \to \widehat{\mathbb{C}}$.

Some classic results on automorphisms of curves of genus $g \geq 2$:

- There is a unique curve having an automorphism of order $4g+2$: Wiman's curve of type I $y^2 = (x^{2g+1} - 1)$, with autom. gr. $G = C_{4g+2}$,
- Except for $g = 3$, there is a unique curve having an automorphism of order $4g$: Wiman's curve of type II $y^2 = x(x^{2g} - 1)$, and autom. gr. $G = C_{4g} \rtimes_{2g-1} C_2$,
  In genus two the group is $G_2 = GL(2,3)$. The exception in genus 3 is Picards's curve $y^3 = (x^4 - 1)$, and gr. $G = C_{12}$
- The largest number,of automorphisms, of type $ag + b$, of a curve in **all** genera is $8g + 8$. For genera $g \equiv 0, 1, 2 \bmod 4$ there is a unique curve: Accola-Maclachlan's curve $y^2 = x(x^{2g+2} - 1)$, and gr. $G = (C_{2g+2} \times C_2) \rtimes C_2$.
  For genera $g \equiv 3 \bmod 4$ there is one more curve: Kulkarni curve $y^{2g+2} = x(x-1)^{g-1}(x+1)^{g+2}$, and gr. $G = \langle x, y : x^{2g+2} = y^4 = (xy)^2 = 1;\ y^2xy^2 = x^{g+2} = 1 \rangle$

Wiman 1895, Accola 1968, Maclachlan 1969, Kulkarni 1991, 1997

Kulkarni showed that, if the Riemann surfaces in a family (of RS of genus g) have more than $4g-4$ automorphisms, then the Teichmüller dimension of the family is 0 or 1.

A finite group G acting on a surface X_g of genus g, $g \geq 2$, is a **large group of automorphisms** of X_g if $|G| \gneq 4g-4$.

Some not that classic results:
Consider families with infinite many genera: $g = p+1$, p prime large enough.

- $g \equiv 2 \mod 3$, there are two surfaces $12(g-1)$ automorphisms, gr. $G = (C_p \rtimes C_6) \times C_2$,
- $g \equiv 2 \mod 3$, there are four surfaces $10(g-1)$ automorphisms, gr. $G = C_p \rtimes C_{10}$,
- $g \equiv 2 \mod 8$, there are two surfaces $8(g-1)$ automorphisms, gr. $G = (C_p \rtimes C_8)$
- All genera: an equisymmetric family of dimension one whose surfaces have $4(g+1)$ automorphisms, gr. $G = D_{g+1} \times C_2$
- All even genera: an equisymmetric family of dimension two whose surfaces have $4(g-1)$ automorphisms, gr. $G = D_{2g-2}$

Belolipetsky-Jones 2005, Costa-I 2018, Reyes-Carocca 2020

Question: What can we say on families of infinite many genera g of Riemann surfaces admitting large groups of automorphisms of order $ag+b$, with a and b integers? (In particular, $g = p+1$, p prime, $b = -a$)

Accola,1995, showed that for primes $p \geq 89$ the only possible orders (at least $8g+8$) are $12(g-1), 10(g-1), 8(g+3)$ and, of course, $8(g+1)$.

Wiman's curves of type II provide examples of surfaces in all genera having $8g$ automorphisms.

Conder-Kulkarni, 1992, provided several families of infinite many genera with Riemann surfaces admitting large groups of automorphisms of order $ag+b \neq \lambda(g-1)$.

Belolipetsky-Jones, 2005, showed the existence of the families of infinite many genera with Riemann surfaces admitting large groups of automorphisms of order $\lambda(g-1)$, with $\lambda \geq 7$, the genus $g = p+1$, where p is a prime $p \geq 17$. They are the only possible surfaces of the corresponding genus admitting at least $7(g-1)$ automorphisms.

One question left: Orders $6(g-1)$ and $5(g-1)$?

Fuchsian Groups

Δ (cocompact) discrete subgroup of $PSL(2,\mathbb{R})$

A (compact) Riemann Surface (Orbifold) of genus $g \geq 2$
$X = \frac{\mathbb{H}}{\Delta}$
Δ has presentation:
generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
relations: $x_i^{m_i}, i = 1 : r,\ x_1...x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
x_i: generator of the maximal cyclic subgroups of Δ
$X = \frac{\mathbb{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
Δ is the orbifold-fundamental group of X.

Area of Δ: area of a fundamental region P

$\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$

X hyperbolic equivalent to $P/\langle\text{pairing}\rangle$

Poincaré's Th: $\Delta = \langle\text{pairing}\rangle$

But from now on $\mu(\Delta) = (2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$, **reduced area**.

Riemann-Hurwitz Formula: If Λ is a subgroup of finite index, N, of a Fuchsian group Δ, then $N = \frac{mu(\Lambda)}{\mu(\Delta)}$

RUT: Any Riemann surface of genus $g \geq 2$ is uniformized by a surface Fuchsian group

$\Gamma_g = \langle a_1, b_1, ..., a_g, b_g \, ; a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}\rangle$

Groups of Automorphisms

G finite group of automorphisms of $X_g = \mathbb{H}/\Gamma_g$, Γ_g a surface Fuchsian group iif there exist
Δ Fuchsian group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$
θ is the monodromy of the regular covering $f : \mathbb{H}/\Gamma_g \to \mathbb{H}/\Delta$

$$\begin{array}{ccc} & \mathbb{H} & \\ \swarrow & & \\ X/ = \mathbb{H}/\Gamma_g \quad & \downarrow & \\ \searrow & & \\ & X/G = \mathbb{H}/\Delta & \end{array}$$

Δ: lifting to $\mathbb{H}$ of G

A morphism $f : X = \mathbb{H}/\Lambda \to Y = \mathbb{H}/\Delta$, X, Y compact Riemann orbifolds, group inclusion $i : \Lambda \to \Delta$
Covering f determined by monodromy $\theta : \Delta \to \Sigma_N$,
$\Lambda = \theta^{-1}(Stb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f $\leftrightarrow$ copy of fund. polygon for Δ)

Teichmüller and Moduli Spaces

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2,\mathbb{R}) \mid \sigma \textit{injective}, \sigma(\Delta) \text{ discrete } \}/PSL(2,\mathbb{R})$

Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h-3+r$, diffeomorphic to a ball of dim $6h-6+2r$.

$\Gamma_g = \pi_1(X)$, surface X of genus g, the **Teichmüller space** is $\mathcal{T}_g := \mathcal{T}_{\Gamma_g}$

The **mapping class group** $M_g^+ = Out(\Gamma_g) = \frac{Diff^+(X)}{Diff_0(X}$

The **moduli space** $\mathcal{M}_g = \mathcal{T}_g/M_(^+ g)$

Mapping class group $M^+(\Delta) = Out(\Delta) = \frac{Diff^+(\mathbb{H}/\Delta)}{Diff_0(\mathbb{H}/\Delta)}$
$\Delta = \pi_1(\mathbb{H}/\Delta)$ as orbifold
$M^+(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta$
$\mathcal{M}_\Delta = \mathcal{T}_\Delta/M^+(\Delta)$

Surfaces with Non-Trivial Automorphisms

If Λ subgroup of Δ ($i:\Lambda\to\Delta$) $\Rightarrow i_*:\mathcal{T}_\Delta\to\mathcal{T}_\Lambda$ embedding

$\Gamma_g\leq\Delta \qquad \mathcal{T}_\Delta\subset\mathcal{T}_g$

G finite group $\qquad \mathcal{T}_g^G=\{[\sigma]\in\mathcal{T}_g\,|\,g[\sigma]=[\sigma]\,\forall g\in G\}\neq\emptyset$

$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Marked surface $\sigma(X)\in\mathcal{T}_g$ and $\beta\in M_g^+$,

$$\begin{array}{ccc} \mathbb{H}/\Delta_g=X & \stackrel{\sigma}{\to} & \sigma(X) \\ \downarrow & & \downarrow \qquad \text{biconformal} \\ \beta_*(X) & \stackrel{\sigma}{\to} & \sigma\beta(X) \end{array}$$

$\beta[\sigma]=[\sigma] \qquad \Leftrightarrow \qquad \gamma\in PSL(2,\mathbb{R}), \quad \sigma(\Gamma_g)=\gamma^{-1}\sigma\beta(\Gamma_g)\gamma$

γ induces an automorphism of the RS $[\sigma(X)]$, $Stb_{\mathcal{M}_g}[\sigma]=Aut([\sigma(X)])$

Action: $\theta:\Delta\to Aut(X_g)=G$, $ker(\theta)=\Gamma_g$

Harvey 1971: $\mathcal{T}_g^G=\bigcup Im(i_*)$, for normal inclusions $i:\Gamma_g\to\Delta$ such that $G\cong\Delta/\Gamma_g$.

For $g\geq 3$ the branch locus of the (orbifold-) universal covering $\mathcal{T}_g\to\mathcal{M}_g$ consists of the RS with non-trivial automorphisms

$Aut(X_g) = G$ conjugate $Aut(Y_g)$ iff $w \in Aut(G), h \in Diff^+(X)$
$\epsilon, \epsilon' : G \to Diff^+(X), \epsilon'(g) = h\epsilon w(g)h^{-1}$
Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv. actions of G iff
θ_1, θ_2 equiv under $Out(\Delta) \times Aut(G)$,
(G, θ), determines the **symmetry** of X
X_g, Y_g equisymmetric if $Aut(X_g)$ conjugate to $Aut(Y_g)$
($Aut(X_g)$: **full automorphism group**)

Broughton (1990): **Equisymmetric Stratification**
$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ is } G\}$.
$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ contains } G\}$.
$\overline{\mathcal{M}}_g^{G,\theta}$ connected, closed alg. var. of $\mathcal{M}_g$ with interior $\mathcal{M}_g^{G,\theta}$.
$\mathcal{M}_g^{G,\theta}$ empty iff $G \neq Aut(X_g)$ for any Riemann surface in $\overline{\mathcal{M}}_g^{G,\theta}$.

Singerman's list of non-maximal signatures. A signature s is called **finitely maximal** if for any Fuchsian group Δ with $s(\Delta) = s$ and a group Δ' containing Δ we have $\dim \mathcal{T}_{\Delta'} < \dim \mathcal{T}_{\Delta}$

$6(g-1)$ Automorphisms

Let $g \geqslant 8$ such that $g-1$ is prime. There exists a compact Riemann surface of genus g with a group of automorphisms of order $6(g-1)$ if and only if $g \equiv 2 \bmod 3$. Moreover, in this case:

(1) the Riemann surfaces form a closed one-dimensional equisymmetric family $\bar{\mathcal{F}}_g$ of Riemann surfaces S with a group of automorphisms G isomorphic to $C_{g-1} \rtimes_6 C_6 = \langle a, c : a^{g-1} = c^6 = 1, cac^{-1} = a^m \rangle$, where m is a primitive 6-th root of unity in the field of $g-1$ elements, and G acts with signature $(0; 2, 2, 3, 3)$,

(2) $\bar{\mathcal{F}}_g$ contains two Riemann surfaces X_1 and X_2 with a group of automorphisms G' of order $12(g-1)$ isomorphic to $(C_{g-1} \rtimes_6 C_6) \times C_2$, acting with signature $(0; 2, 6, 6)$, and

(3) if $S \in \mathcal{F}_g$, where $\mathcal{F}_g$ the interior of $\bar{\mathcal{F}}_g$, then G is the full automorphism group of S, and

(4) if $g > 14$ then the subset $\bar{\mathcal{F}}_g \setminus \mathcal{F}_g$ of $\bar{\mathcal{F}}_g$ is $\{X_1, X_2\}$, with full automorphism group G'.

$5(g-1)$ Automorphisms

Let $g \geqslant 8$ such that $g-1$ is prime. There exists a compact Riemann surface S of genus g with a group of automorphisms G of order $5(g-1)$ if and only if $g \equiv 2 \bmod 5$. Moreover, in this case:

(1) the group G is isomorphic to

$$C_{g-1} \rtimes_5 C_5 = \langle a, b : a^{g-1} = b^5 = 1, bab^{-1} = a^r \rangle,$$

where r is a primitive 5-th root of unity in the field of $g-1$ elements, and G acts with signature $(0; 5, 5, 5)$,

(2) the action of G extends to an action of a group G' isomorphic to

$$C_{g-1} \rtimes_{10} C_{10} = \langle a, c : a^{g-1} = c^{10} = 1, cac^{-1} = a^{-r} \rangle,$$

with r as before, and G' acts with signature $(0; 2, 5, 10)$,

(3) there are exactly four pairwise non-isomorphic such Riemann surfaces S,

(4) the full automorphism group of S is G', and

Remarks

1. For $g \geqslant 8$, the two surfaces X_1, X_2 above are the two surfaces with $12(g-1)$ automorphisms obtained by Belolipetsky-Jones (2005).

2. For $g = 14$, there are three other surfaces in $\bar{\mathcal{F}}_{14}$ with automorphism group $PSl(2, 13)$. These surfaces were obtained by Macbeath (1968), also Belolipetsky-Jones (2005).

3. The four surfaces admitting $5(g-1)$ automorphisms are in fact the four surfaces obtained by Belolipetsky-Jones (2005) with exactly $10(g-1)$ automorphisms.

4. For genera $g \geqslant 8$ such that $g-1$ is prime. There exists a compact Riemann surface S of genus g with a group of automorphisms of order $3(g-1)$ if and only if $g \equiv 2$ mod 3. Furthermore, in this case the Riemann surface belongs to the family $\bar{\mathcal{F}}_g$ above. As a consequence, there are no compact Riemann surfaces of genus g with full automorphism group of order $3(g-1)$.

$6(g-1)$ Automorphisms

S, a compact Riemann surface of genus $g \geqslant 8$, where $p = g-1$ is prime, and G a group of automorphism of order $6q$.

1. By the Riemann–Hurwitz formula the possible signatures of the action of G on S are $\sigma_1 = (0;2,2,3,3), \sigma_2 = (0;2,2,2,6)$ and $\sigma_3 = (0;3,6,6)$ for each genus and, in addition, the signature $(0;2,7,42)$ for $g=8$; but there is not surface epimorphism $\theta\Delta(0;2,7,42) \to C_{42}$.

2. By Sylow's Theorems, $G = C_p \times D_3, D_{3p}, C_{6p}, C_p \rtimes_2 C_6, C_p \rtimes_3 C_6, C_p \rtimes_6 C_6$. But there are no surface epimorphisms from Fuchsian groups with signatures $\sigma_1 = (0;2,2,3,3), \sigma_2, \sigma_3$ onto $C_p \times D_3, D_{3p}, C_{6p}$, nor $C_p \rtimes_2 C_6$. There are no surface epimorphisms from a Fuchsian group $\Delta(0;2,2,2,6)$ onto $C_p \rtimes_3 C_6$ or $C_p \rtimes_6 C_6$.

3. There is no surface epimorphism $\theta : \Delta(0;2,2,3,3) \to C_p \rtimes_3 C_6$. But there are surface epimorphisms $\theta\Delta(0;2,2,3,3) \to C_p \rtimes_6 C_6 = \langle a,b,s : a^q = b^3 = s^2 = 1, [s,b] = 1, bab^{-1} = a^r, sas = a^{-1}\rangle$, and r a primitive third root of unity in $\mathbb{F}_q$.

A surface epimorphism $\theta_3 : \Delta(0;2,2,3,3) \to C_q \rtimes_6 C_6$ is equivalent to one of the form:

$$\theta_{3,m}(x_1) = s,\ \theta_{3,m}(x_2) = as,\ \theta_{3,m}(x_3) = a^{1+(1+r)m}b^2,\ \theta_{3,m}(x_4) = a^m b,\ 1 \leqslant m \leqslant q$$

4 Iterating a suitable number of times the braid $\Phi^2_{3,4}$, each epimorphism $\theta_{3,m}$ is equivalent to $\theta_{3,0}$

$$\theta_{3,0}(x_1) = s,\ \theta_{3,0}(x_2) = as,\ \theta_{3,0}(x_3) = ab^2 \text{ and } \theta_{3,0}(x_4) = b$$

Then $\bar{\mathcal{F}}_g$ is an equisymmetric family with non-empty interior. Otherwise by Singerman (1972) the action with monodromy $\theta_{3,0}$ extends to an action with monodromy $\widehat{\theta} : \Delta(0; 2, 2, 2, 3) \to G_{12p}$. But this action does not exist by Belolipetsky-Jones (2005).

5 There are surface epimorphisms from $\Delta_1(0; 3, 6, 6)$ onto both $C_q \rtimes_3 C_6$ and $C_q \rtimes_6 C_6$. They are equivalent to one defined by $\theta_{1,i}(x_1) = b^i,\ \theta_{1,i}(x_2) = a^{-r^i} b^i s$, $\theta_{1,i}(x_3) = a^i bs$, $i = 1, 2$ onto $C_q \rtimes_3 C_6$, or to the one defined by $\theta_2(x_1) = ab,\ \theta_2(x_2) = bs$, $\theta_2(x_3) = a^r bs$. onto $C_q \rtimes_6 C_6$.

6 By Singerman (1972) and Belolipetsky-Jones (2005) these monodromies could extend only to monodromies $\Theta : \Delta_2(0; 2, 6, 6) = \langle y_1, y_2, y_3 \mid y_1^2 = y_2^6 y_3^6 y_1 y_2 y_3 = 1\rangle \to (C_p \rtimes_6 C_6) \times C_2 = \langle a, b, s\rangle \times \langle z\rangle$.

The Belolipetsky-Jones Exceptional Surfaces X_1, X_2

7 According to Belolipetsky-Jones (2005), for primes $p > 13$, there are just two surfaces X_1, X_2 of genus $g = p + 1$ admitting exactly $12p$ automorphisms. The surfaces are determined by two non-equivalent actions of $(C_p \rtimes_6 C_6) \times C_2$ with monodromies $\Theta_1(y_1) = as,\ \Theta_1(y_2) = b^2sz,\ \Theta_1(y_3) = a^{-r}bz$, and $\Theta_2(y_1) = as,\ \Theta_2(y_2) = bsz,\ \Theta_2(y_3) = a^{-r^2}b^2z$ respectively.

8 The monodromies $\theta_{1,i}, \theta_2$ DO extend.
Firstly, setting $x_1' = y_2^2, x_2' = y_3$ and $x_3' = (y_2^2y_3)^{-1}$, the restrictions $\Theta_1|_{\langle x_1',x_2',x_3'\rangle}, \Theta_2|_{\langle x_1',x_2',x_3'\rangle} : \Delta_1 \cong \langle x_1', x_2', x_3'\rangle \to C_p \rtimes_3 C_6$ are precisely $\theta_{1,1}$ and $\theta_{1,2}$ respectively.
Secondly, setting $x_1'' = y_3^2, x_2'' = y_2$ and $x_3'' = (y_3^2y_2)^{-1}$, the restrictions $\Theta_1|_{\langle x_1'',x_2'',x_3''\rangle}, \Theta_2|_{\langle x_1'',x_2'',x_3''\rangle} : \Delta_1 \cong \langle x_1'', x_2'', x_3''\rangle \to C_q \rtimes_6 C_6$ are equivalent to θ_2.
It follows that a Riemann surface S with the action θ_2 of $C_q \rtimes_6 C_6$ with signature $(0; 3, 6, 6)$ is isomorphic to either X_1 or X_2.

9 Finally X_1, X_2 belong to $\bar{\mathcal{F}}_g$ since $\hat{x}_1 = (y_1y_2^2y_3^2)^{-1}, \hat{x}_2 = y_1, \hat{x}_3 = y_2^2$ and $\hat{x}_4 = y_3^2$ generate a subgroup of Γ_2 isomorphic to $\Delta(0; 2, 2, 3, 3)$. Furthermore, the restrictions $\Theta_1|_{\hat{\Gamma}}$ and $\Theta_2|_{\hat{\Gamma}}$ are monodromies equivalent to $\theta_{3,0}$.

THE END