Johnson morphisms: definitions and related open questions

Rachel Davis
University of Wisconsin-Madison

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Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups, and Related Topics
S. Allen Broughton, Jen Paulhus, Aaron Wootton

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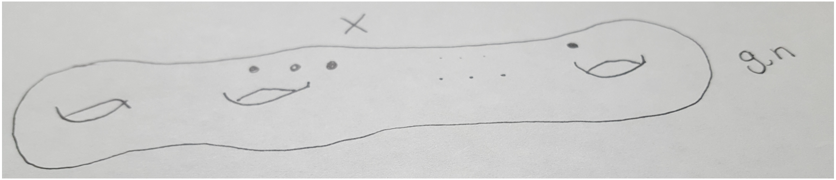
Let X be a complex curve of genus g with n punctures (i.e. a Riemann surface) defined over a number field K.



Fix ℓ a prime. Write H for the Tate module $T_\ell \mathrm{Jac}(X)$. This is a rank $2g$ $\mathbb{Z}_\ell$-module with an action of the absolute Galois group G_K.

Definition

The **mapping class group**, $\mathcal{M}_{g,1}$ of X, is the group of isotopy classes of homeomorphisms, of X inducing the identity on the boundary.

- The mapping class group can be embedded in the automorphism group of a free group on $r = 2g$ generators by work of Dehn and Nielsen.
- The Johnson homomorphism has been used to investigate the mapping class group in $\mathrm{Aut}(F_r)$. (Birman, Kitano, Morishita, Perron, Putman, $\ldots$)

- Let Γ be the pro-ℓ completion of the étale fundamental group of $X_{\overline{K}}$.

- Let $F = F_r$ be the free pro-ℓ group on $r = 2g$ generators.

- Let $\{\Gamma_k\}$ denote the lower central series of Γ, defined by $\Gamma_k = \overline{[\Gamma, \Gamma]_{k-1}}$ is the closure of the subgroup generated by commutators of elements of Γ with elements of Γ_{k-1}.

- Then, $H = F^{\text{ab}}$. Let $[f] := f \pmod{[\Gamma, \Gamma]}$.

- Let $T = T(H)$ be the tensor algebra on H,

$$T := {}_{m \geq 0} H^{\otimes m},$$

which can be identified with the $\mathbb{Z}_\ell$-algebra

$$\mathbb{Z}_\ell\langle\langle X_1, \ldots X_r \rangle\rangle$$

of non-commutative power series, where $X_j = [x_j]$ $(1 \leq j \leq r)$.

- Let $T_n := \prod_{m \geq n} H^{\otimes m}$ be the two-sided-ideal of T consisting of power series of degree $\geq n$.
- A $\mathbb{Z}_\ell$-algebra automorphism ϕ of T is called filtration-preserving if $\phi(T_n) = T_n$ for all $n \geq 0$.
- Denote by $\mathrm{Aut}^{\mathrm{fil}}(T)$ the group of filtration-preserving $\mathbb{Z}_\ell$ automorphisms of T.

- Note that the homomorphism

$$\mathrm{Aut}^{\mathrm{fil}}(T) \to \mathrm{GL}(H)$$

splits.
- The splitting $\iota : \mathrm{GL}(H) \to \mathrm{Aut}^{\mathrm{fil}}(T)$ is given by

$$\iota([\phi])(t_m) = ([\phi]^{\otimes m}(t_m))$$

for $t_m \in H^{\otimes m}$.

Definition

The group

$$\mathrm{IA}(T) := \mathrm{Ker}(\mathrm{Aut}^{\mathrm{fil}}(T) \to \mathrm{GL}(H))$$

is called the **Torelli subgroup**.

Also,

$$\mathscr{I}_{g,1} := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{GL}(H))$$

is called the **Torelli group** of X.

Lemma

- $\mathrm{Aut}^{fil}(T) \simeq \mathrm{IA}(T) \rtimes \mathrm{GL}(H)$ *given by* $\phi \mapsto (\phi \circ [\phi]^{-1}, [\phi])$.
- *There is a bijection* $\mathrm{IA}(T) \simeq \mathrm{Hom}(H, T_2)$ *given by* $\phi \mapsto \phi\,|_H - \mathrm{id}_H$.

Let $\mathbb{Z}_\ell[[\Gamma]]$ be the complete group algebra of Γ over $\mathbb{Z}_\ell$.

Definition

The **Magnus expansion** $\theta : F \hookrightarrow T^\times$ defined by $\theta(x_j) = 1 + X_j$ is extended to a $\mathbb{Z}_\ell$-algebra isomorphism $\hat{\theta} : \mathbb{Z}_\ell[[\Gamma]] \simeq T$.

Then, the extended pro-ℓ Johnson homomorphism is defined by

$$\hat{\tau} : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}^{\mathrm{fil}}, \hat{\tau}(\phi) := \hat{\theta} \circ \hat{\phi} \circ \hat{\theta}^{-1}.$$

If we let $(\tau(\phi), [\phi])$ be the corresponding pair in $\mathrm{IA}(T) \rtimes \mathrm{GL}(H)$, then $\tau : \mathrm{Aut}(\Gamma) \to \mathrm{IA}(T)$.

Composing τ with $\mathrm{Aut}_k(\Gamma) \to \mathrm{Hom}(H, T_2) \to \mathrm{Hom}(H, H^{\otimes m})$ yields the m^{th} pro-ℓ Johnson map

$$\tau_m : \mathrm{Aut}(F) \to \mathrm{Hom}(H, H^{\otimes m}).$$

Let q be the symplectic element of $\wedge^2 H$.

Let $f : [\Lambda^3 H] \to H \otimes [(\wedge^2 H)/q]$ defined by

$$a \wedge b \wedge c \mapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b).$$

Theorem (Johnson)

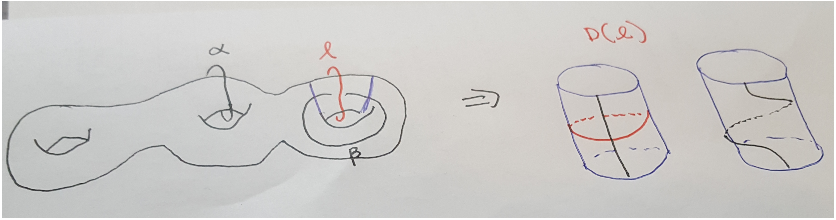
Let $g \geq 2$. The image of τ_2 on $\mathscr{I}_{g,1}$ lies in the image of f.

- When restricted to the Torelli subgroup, $\tau_2 : \mathscr{I}_{g,1} \to \wedge^3 H$ is a homomorphism.

- This homomorphism extends to a homomorphism $\bar{\tau}_2 : \mathcal{M}_{g,1} \to \left(\frac{1}{2} \wedge^3 H\right) \rtimes \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.

- The map $\bar{\tau}_2$ also corresponds to a crossed homomorphism $\varphi \in H^1(\mathcal{M}_{g,1}, \frac{1}{2} \wedge^3 H)$.

The mapping class group of a closed genus g Riemann surface is generated by **Dehn twists** around simple closed curves.



Dehn twist around ℓ which acts trivially on α and nontrivially on β.

The action of $D(x_i)$ on Γ is the following:
$D(x_i)(y_i) = y_i x_i$ and $D(x_i)(z_j) = z_j$ for $z_j \neq y_i$.

Example (Perron)

Let $g = 2$ and $C_1 = x_2 y_2^{-1} x_2^{-1} y_2 \in \Gamma$.
Then,

$$D(C_1)(x_1) = x_1 y_1^{-1} x_2 y_2 x_2^{-1},$$

$$D(C_1)(x_2) = C_1 x_2,$$

$$D(C_1)(y_1) = C_1 y_1 C_1^{-1}, \text{ and}$$

$$D(C_1)(y_2) = y_2.$$

Fox calculus Let Γ be a free group generated by $z_1, \ldots, z_n$. Let $\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ denote the evaluation map defined by

$$\varepsilon\left(\sum_i n_i g_i\right) = \sum_i n_i.$$

Definition

We define partial derivatives $\frac{\partial}{\partial z_i} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ by

$$\frac{\partial z_j}{\partial z_i} = \delta_{ij}, \quad \frac{\partial(u+v)}{\partial z_i} = \frac{\partial u}{\partial z_i} + \frac{\partial v}{\partial z_i}, \quad \frac{\partial(uv)}{\partial z_i} = \varepsilon(v)\frac{\partial u}{\partial z_i} + u\frac{\partial v}{\partial z_i}.$$

As a consequence, we have $\frac{\partial u^{-1}}{\partial z_i} = -u^{-1}\frac{\partial u}{\partial z_i}$.

Example

$$\frac{\partial(ab)}{\partial a} = 1,\ \frac{\partial(ab)}{\partial b} = a,\ \frac{\partial(ba)}{\partial a} = b,\ \frac{\partial(ba)}{\partial b} = 1.$$

Example

$$\frac{\partial(x_1 y_1 x_1^{-1})}{\partial y_1} = x_1 \quad \text{and} \quad \frac{\partial(x_1 y_1 x_1^{-1})}{\partial x_1} = 1 - x_1 y_1 x_1^{-1}.$$

Definition

The Fox matrix of $f \in \mathcal{M}_{g,1}$ is the $2g \times 2g$ matrix with coefficients in $\mathbb{Z}[\Gamma]$ defined by

$$B(f) = \begin{pmatrix} \overline{\frac{\partial f(z_1)}{\partial z_1}} & \cdots & \overline{\frac{\partial f(z_{2g})}{\partial z_1}} \\ \overline{\frac{\partial f(z_1)}{\partial z_2}} & \cdots & \overline{\frac{\partial f(z_{2g})}{\partial z_2}} \\ \vdots & & \vdots \\ \overline{\frac{\partial f(z_1)}{\partial z_{2g}}} & \cdots & \overline{\frac{\partial f(z_{2g})}{\partial z_{2g}}} \end{pmatrix}$$

where $\overline{()}$ is the anti-isomorphism $\overline{\sum_i n_i g_i} = \sum_i n_i g_i^{-1}$.

To each $\alpha \in \mathbb{Z}[\Gamma]$, associate a formal series in $F(u_1, \ldots, u_g, v_1, \ldots v_g)$ where u_i (respectively v_i) corresponds to $x_i - 1$ (resp. $y_i - 1$).

Doing this, we can associate

$$B(f) = B_0(f) + \cdots B_k(f) + \cdots$$

where $B_k(f)$ is a $2g \times 2g$ matrix with entries in I_k, the abelian group generated by $\{w_{j_k} w_{j_{k-1}} \cdots w_{j_1}; 1 \leq j_i \leq 2g\}$ where $w_i = u_i$ if $1 \leq i \leq g$ and $w_i = v_{i-g}$ if $g < i \leq 2g$.

The entry $a_{ij}^{(k)}$ of $B_k(f)$ is given by

$$a_{ij}^{(k)} = \sum_{1 \leq j_k, \cdots, j_1 \leq 2g} \frac{\partial^k}{\partial z_{j_k} \cdots \partial z_{j_1}} \left(\overline{\frac{\partial f(z_j)}{\partial z_i}} \right) (1) w_{j_k} \cdots w_{j_1}.$$

For $f \in \mathcal{M}_{g,1}$, set

$$A_k(f) = B_k(f) \times B_0(f)^{-1} \in \mathcal{M}_{2g}(I_k).$$

Then,

$$A_k : \mathcal{M}_{g,1} \to \mathcal{M}_{2g}(I_k) \simeq I_k \otimes H \otimes H \simeq (\otimes^k H) \otimes H \otimes H.$$

Let $\widetilde{A_1}$ be the map obtained by composing A_1 with the projection map $\pi : \otimes^3 H \to \wedge^3 H$. Then $\widetilde{A_1}$ is a crossed homomorphism.

Corollary (Johnson)

Restrict $\widetilde{A_1}$ to the Torelli group, $\mathscr{I}_{g,1}$. Then, it is equal to $-6\tau_2$, where τ_2 is the second Johnson homomorphism.

Example (Perron)

Let $g = 2$ and $C_1 = x_2 y_2^{-1} x_2^{-1} y_1 \in \Gamma$. Then, $D(C_1)(x_1) = x_1 y_1^{-1} x_2 y_2 x_2^{-1}$, $D(C_1)(x_2) = C_1 x_2$, $D(C_1)(y_1) = C_1 y_1 C_1^{-1}$, $D(C_1)(y_2) = y_2$. Then,

$$A_1(D(C_1)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_2 & -b_1 & 0 & 0 \\ a_1 - b_1 & b_2 & b_2 & 0 \\ -a_1 + b_1 - a_2 & a_2 - b_2 & -b_1 & 0 \end{pmatrix}.$$

$$M_0 = \begin{pmatrix} a_1 \otimes b_1 & a_1 \otimes b_2 & a_1 \otimes -a_1 & a_1 \otimes -a_2 \\ a_2 \otimes b_1 & a_2 \otimes b_2 & a_2 \otimes -a_1 & a_2 \otimes -a_2 \\ b_1 \otimes b_1 & b_1 \otimes b_2 & b_1 \otimes -a_1 & b_1 \otimes -a_2 \\ b_2 \otimes b_1 & b_2 \otimes b_2 & b_2 \otimes -a_1 & b_2 \otimes -a_2 \end{pmatrix}$$

Example (Perron)

Then,

$$\widetilde{A_1}(D(C_1)) = b_2 \wedge a_2 \wedge b_1 - b_1 \wedge a_2 \wedge b_2 + b_2 \wedge b_1 \wedge -a_1$$

$$-a_1 \wedge b_2 \wedge b_1 - a_2 \wedge b_2 \wedge b_1 - b_1 \wedge b_2 \wedge -a_1$$

$$= 3(a_1 + a_2) \wedge b_1 \wedge b_2.$$

Morita extended the Johnson homomorphism τ_2 to a crossed homomorphism φ. Let

$$\widetilde{A_{10}} = \widetilde{A_1} \rtimes B_0 : \mathcal{M}_{g,1} \to (3 \wedge^3 H) \rtimes \mathrm{Sp}(2g, \mathbb{Z}_\ell).$$

Theorem (Morita)

$\widetilde{A_{10}}$ and $\varphi|_{\mathcal{M}_{g,1}}$ define the same class in $H^1(\mathcal{M}_{g,1}, \frac{1}{2} \wedge^3 H)$.

Choose a point $P \in X(K)$. The Abel-Jacobi map is a map

$$X \to \mathrm{Jac}(X),$$

given by $Q \mapsto [Q - P]$. Let $[X] = [Q - P]$ and let $[X^-] = i_*[X]$ where i is the involution of $\mathrm{Jac}(X)$ mapping each point to its inverse. The Ceresa cycle is defined as $Z = [X] - [X^-] \in \mathrm{CH}_1(\mathrm{Jac}(X))$.

For a hyperelliptic curve, the Ceresa cycle is algebraically equivalent to zero since the action of the hyperelliptic involution agrees with negation on the Jacobian. All curves of genus 2 are hyperelliptic.

Hain and Matsumoto relate the Ceresa cycle to the images of Galois representations by studying its image, the Ceresa class, $\nu(X) \in H^1(G_K, L)$, where L is the quotient of $\wedge^3 H$ by $H \wedge q$ where q is the symplectic form.

Theorem (Hain-Matsumoto, 2005)

Let $g \geq 3$. Let $m(X) \in H^1(G_K, L)$ be the pullback of $\widetilde{A_{10}} \in H^1(\mathcal{M}_{g,1}, L)$ along the homomorphism

$$\rho_X^{(\ell)} : G_K \to \mathcal{M}_{g,1}$$

Then,

$$\nu(X) = m(X).$$

Theorem (Ceresa, 1983)

On a generic Jacobian variety $\mathrm{Jac}(X)$ of dimesion $g \geq 3$, the Ceresa cycle is not algebraically trivial.

Open questions: Are there non-hyperelliptic curves (necessarily $g \geq 3$) such that $\nu(X) = 0$? What are examples?

Theorem (Bisogno-Li-Litt-Srinivasan)

Let X be the unique genus 7 Hurwitz curve (non-hyperelliptic).

$$1+7xy+21x^2y^2+35x^3y^3+28x^4y^4+2x^7+2y^7=0$$

(affine plane model given by Brock). Then $\nu(X)$ is torsion.

Theorem (Corey-Ellenberg-Li)

Let X be a smooth, projective curve over $\mathbb{C}((t))$ with semistable reduction at the special fiber. Then the Ceresa class $\nu(X)$ is torsion.

Questions?