One Dimensional Equisymmetric Strata in Moduli Space

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Moduli spaces. Equisymmetric stratification

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$$\mathcal{M}_g$$

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- Structures of Riemann surfaces on surfaces of genus g:

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- $\mathcal{M}_g$ admits a stratification into a finite, disjoint union of *equisymmetric strata*.
- Each stratum corresponds to a collection of surfaces whose automorphism groups are *topologically equivalent*.

Strata

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- The zero dimensional strata correspond to the well-studied quasi-platonic surfaces.
- At the other extreme are the open, dense stratum of surfaces with no automorphisms, of dimension $3g - 3$, and the hyperelliptic locus of dimension $2g - 1$.
- As the genus increases there is an explosive growth in the number of strata in the intermediate dimensions. The topology of these individual strata is largely unknown.

One dimensional strata

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- We describe those strata of dimension one, as punctured Riemann surfaces, in terms of the data of the automorphism group.

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- 1) interior punctures corresponding to surfaces with exceptional symmetries
- 2) punctures at infinity corresponding to limiting nodal Riemann surfaces (in the compactification of $\mathcal{M}_g$)

Cases

Let $\mathfrak{S}$ be a stratum of dimension one and $S \in \mathfrak{S}$. There are two types of strata:

- **Case 1:** $S/\mathrm{Aut}(S)$ is the sphere and $S \to S/\mathrm{Aut}(S)$ is branched over four points.

Cases

Let $\mathfrak{S}$ be a stratum of dimension one and $S \in \mathfrak{S}$. There are two types of strata:

- **Case 1:** $S/\mathrm{Aut}(S)$ is the sphere and $S \to S/\mathrm{Aut}(S)$ is branched over four points.
- **Case 2:** $S/\mathrm{Aut}(S)$ is a torus and $S \to S/\mathrm{Aut}(S)$ is branched over one point.

Coverings and group actions

The quotient surface $S/G = T$ of a conformal action of a finite group G is a closed Riemann surface of genus h with a unique conformal structure so that

$$\pi_G : S \to S/G = T$$

is holomorphic.

- The quotient map $\pi_G : S \to T$ is branched over a finite set B_G such that π_G is an unbranched covering over $T^\circ = T - B_G$. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unbranched covering whose group of deck transformation equals G, restricted to S°.

Coverings and group actions

The covering $\pi_G : S^\circ \to T^\circ$ determine a normal subgroup $\Pi_G = \pi_1(S^\circ) \lhd \pi_1(T^\circ)$ and an exact sequence:

$$\Pi_G \hookrightarrow \pi_1(T^\circ) \stackrel{\xi}{\twoheadrightarrow} G.$$

- **Note**: If $\alpha \in \mathrm{Aut}(G)$ then $\xi' = \alpha \circ \xi$ determines the same kernel and hence the constructed surfaces are the same: $S^\circ \subset S$ lying over $T^\circ \subset T$ and the G-actions upon them are equivalent.

Classes of generating vectors (notations)

- $$K_G(h : g_1^G, \ldots, g_t^G) =$$
$$\{(a_1, \ldots, a_h, b_1, \ldots, b_h, c_1, \ldots, c_t) : \prod_{i=1}^{h}[a_i, b_i]\prod_{j=1}^{t} c_j = 1, c_i \in g_i^G\}$$
$$\text{where } g^G = \{g^a : a \in G\}$$

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- $$K_G^\circ(h : g_1^G, \ldots, g_t^G) = \{\text{vectors in } K_G(h : g_1^G, \ldots, g_t^G) \text{ generating } G\}$$

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- $$\widetilde{K_G^{\circ}}(h : g_1^G, \ldots, g_t^G) = \mathrm{Aut}(G) - \text{classes in } K_G^{\circ}(h : g_1^G, \ldots, g_t^G)$$

The quotient surface $S/G = \widehat{\mathbb{C}}$ is the Riemann sphere with

$$\pi_G : S \to S/G = \widehat{\mathbb{C}}$$

holomorphic.

- The quotient map $\pi_G : S \to \widehat{\mathbb{C}}$ is branched over a four points $\{z_1, z_2, z_3, z_4\}$. The covering $\pi_G : S^\circ = S - \pi_G^{-1}\{z_1, z_2, z_3, z_4\} \to \widehat{\mathbb{C}} - \{z_1, z_2, z_3, z_4\}$ is given by a monodromy $\pi_1(\widehat{\mathbb{C}} - \{z_1, z_2, z_3, z_4\}) \to G$.

Case1: Subcases

The signature is $(0; n_1, \ldots, n_4)$ (we shall denote by $(n_1, \ldots, n_4)$) and the G-signature is $(c_1^G, \ldots, c_4^G)$ for some generating vector $(c_1, \ldots, c_4)$, with $c_i^{n_i} = 1$.
Since is $g \geq 2$ then the case $(n_1, \ldots, n_4) = (2, 2, 2, 2)$ is excluded.
Subcases:

1. Pure braid case: all the c_i^A are distinct.

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Subcases:

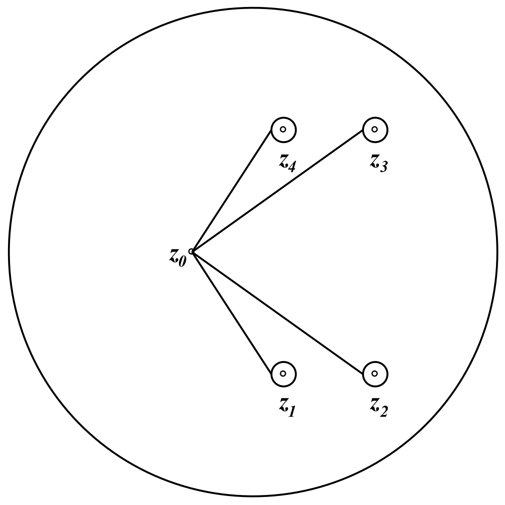
1. Pure braid case: all the c_i^A are distinct.
2. Non-pure braid case: at least two of the c_i^A are equal. This case will be derived from the pure braid case.

Pure braid subcase

Recall that $T^{\circ} = \widehat{\mathbb{C}} - \{z_1, z_2, z_3, z_4\}$ we have

$$\pi_1(T^{\circ}) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_1\gamma_2\gamma_3\gamma_4 = 1 \rangle$$

The loop γ_i issues from a base point z_0, encircles z_i in a small counterclockwise loop and returns to z_0 along the original path. The γ_i issue from z_0, in distinct directions in cyclic counterclockwise order.


z4
z3
z0
z1
z2

By the Möbius transformation $L(z) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3}$, we can consider:

$$\pi_{G,\lambda} : S \to S/G \xrightarrow{L} \widehat{\mathbb{C}}$$

is branched over $\{0, 1, \infty, \lambda\}$.

Let $T^\circ = \mathbb{C}_\lambda = \mathbb{C} - \{0, 1, \lambda\} = \widehat{\mathbb{C}} - \{0, 1, \lambda, \infty\}$. The conformal structure of $\mathbb{C}_\lambda$ is given by $\lambda \in B = \mathbb{C} - \{0, 1\} = \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

Action of fundamental group

Let $\left(c_1^G, \ldots, c_4^G\right)$ be a G-signature.
There is an action of $\pi_1(B)$ on $K_G^\circ(c_1^G, \ldots, c_4^G)$

- $\pi_1(B, \lambda)$ is generated by counterclockwise loops β_0, β_1 y β_∞ around 0, 1, ∞, respectively, with product 1.

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- Let $(\beta_0)_*$ be the Denh twist around β_0 (with the orientation given by β_0. Similarly for $(\beta_1)_*$ and $(\beta_\infty)_*$.

Action on generator systems

By conjugation the three following operations may be expresed as:

	γ_1	γ_2	γ_3	γ_4
$(\beta_0)_*$	γ_1	$\gamma_2^{\gamma_4}$	$\gamma_3^{\gamma_4}$	$\gamma_4^{\gamma_1^{-1}}$
$(\beta_1)_*$	γ_1	$\gamma_2^{\gamma_1^{-1}\gamma_4^{-1}\gamma_1}$	γ_3	$\gamma_4^{\gamma_1\gamma_3}$
$(\beta_\infty)_*$	γ_1	γ_2	$\gamma_3^{\gamma_4^{-1}\gamma_3^{-1}}$	$\gamma_4^{\gamma_3^{-1}}$

where: $\gamma^w = w\gamma w^{-1}$.

The stratum for this subcase

Let $\mathcal{O}$ be an orbit of $\pi_1(B)$ on the $\mathrm{Aut}(G)$-classes $\widetilde{K_G^\circ}(g_1^G, \ldots, g_4^G)$, and let $\{\mathcal{O}_{0,i} : i\}$, $\{\mathcal{O}_{1,j} : j\}$, $\{\mathcal{O}_{\infty,k} : k\}$ be the orbit decomposition of $\mathcal{O}$ with respect to the cyclic subgroups $\langle\beta_{0*}\rangle$, $\langle\beta_{1*}\rangle$, $\langle\beta_{\infty*}\rangle$.

There is a Riemann surface $\mathcal{S}$ which is an unbranched covering $\mathcal{S} \rightarrow B$, such that $\mathcal{S} - \{p_1, ..., p_s\}$ is an **one dimensional stratum**.

The covering surface $\mathcal{S}$ and the covering are completely described by the following:

The stratum for this subcase

Theorem

1. *The degree of the covering $\mathcal{S} \to B$ is the size of the orbit $|\mathcal{O}| = m$.*
2. *Let $\mathcal{O} = \{o_1, ..., o_m : o_i \in \widetilde{K_G^\circ}(g_1^G, \ldots, g_4^G)\}$. The monodromy of the covering $\mathcal{S} \to B$ is*

$$\omega : \pi_1(B) \to \Sigma_{|\mathcal{O}|} = \mathcal{P}\{1, ..., m\}$$

defined by $\omega(\beta)(i) = j$ if for each $(c_1, ..., c_4) \in o_i$ we have that $\beta_*(c_1, ..., c_4) \in o_j$, where $\beta \in \pi_1(B)$.

The stratum for this subcase

Theorem

3. The sets of local degrees above $0, 1, \infty$ *are the sets* $\{|\mathcal{O}_{0,i}| : i\}$, $\{|\mathcal{O}_{1,j}| : j\}$, $\{|\mathcal{O}_{\infty,k}| : k\}$, *respectively.*

4. The Riemann surface S *is a surface of genus*

$$\rho = 1 + \frac{(h_0 + h_1 + h_\infty) - n}{2}$$

with $h_0 + h_1 + h_\infty$ *punctures, where* h_0, h_1, h_∞ *are the number of orbits in* $\{\mathcal{O}_{0,i} : i\}$, $\{\mathcal{O}_{1,j} : j\}$, $\{\mathcal{O}_{\infty,k} : k\}$ *respectively.*

Anharmonic group

We now consider the possibilities that not all the c_i^G are distinct.
Anharmonic group:

Permutations	λ'
$id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)$	λ
$(1,2), (3,4), (1,4,2,3), (1,3,2,4)$	$1-\lambda$
$(1,3), (2,4), (1,2,3,4), (4,3,2,1)$	$1/\lambda$
$(1,4), (2,3), (1,3,4,2), (1,2,4,3)$	$\lambda/(\lambda-1)$
$(1,2,3), (4,3,2), (4,2,1), (1,3,4)$	$(\lambda-1)/\lambda$
$(3,2,1), (2,3,4), (1,2,4), (4,3,1)$	$1/(1-\lambda)$

Subcase 2

We now consider the possibilities that not all the c_i^G are distinct. We cover one case: we consider the case the $c_1^G = c_2^G$, but $c_3^G \neq c_4^G$ and neither of c_3^G, c_4^G equal c_1^G.

- We start as in the case of the pure braid group action. We obtain an unbranched covering $\mathcal{S} \to B$ given by a monodromy $\omega : \pi_1(B) \to \Sigma_{|\mathcal{O}|}$.

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- We start as in the case of the pure braid group action. We obtain an unbranched covering $\mathcal{S} \longrightarrow B$ given by a monodromy $\omega : \pi_1(B) \longrightarrow \Sigma_{|\mathcal{O}|}$.
- We consider the order 2 subgroup $\langle 1 - \lambda \rangle$ of the anharmonic group. If $(1 - \lambda)_*(\ker \omega) = \ker \omega$, then $(1 - \lambda)$ lifts to $\mathcal{S}$ and $\mathcal{S}/\widetilde{(1 - \lambda)}$ is the surface containing the stratum. If $(1 - \lambda)_*(\ker \omega) \neq \ker \omega$ then $\mathcal{S}$ is the stratum.

Case 2

The quotient surface $S/G = T$ is a torus with

$$\pi_G : S \to S/G = T$$

holomorphic.

- The quotient map $\pi_G : S \to T$ is branched over a point p. The covering $\pi_G : S^\circ = S - \{\pi_G^{-1}(p)\} \to T^\circ = T - \{p\}$ is given by a monodromy $\pi_1(T - \{p\}) \to G$.

Vectors

The signature is $(1; n)$ and G-signatures for this case are (g^G) with $g^n = 1$.

$$K_G(1 : g^G) = \{(a, b, x) : [a, b]x = 1, x \in g^G\}$$

$$K_G^\circ(1 : g^G) = \{\text{vectors in } K_G(1 : g^G) \text{ generating } G\}$$

$$\widetilde{K_G^\circ}(h : g_1^G, \ldots, g_t^G) \ \mathrm{Aut}(G) - \text{classes in } K_G^\circ(h : g_1^G, \ldots, g_t^G)$$

Subcase 1

Let S be a surface of the stratum that we want to study. Assume the monodromy of $\pi_G : S \to T$ is:

$$\xi : \pi_1(T - \{p\}) = \langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1 \rangle \to G.$$

The group G has an automorphism $G \to G$ given by:

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1} \quad \xi(\beta) \longmapsto \xi(\beta)^{-1}$$

- In this case the group G is not the full group of automorphisms of the surfaces of the strata. There is a group $H \geq G$, such that the action of H on the surfaces of the strata have signature $(2, 2, 2, 2n)$, and then we are in the Case 1: orbit space of genus 0.

Subcase 2.

- There is not an automorphism $G \to G$ given by:

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1} \quad \xi(\beta) \longmapsto \xi(\beta)^{-1}$$

Example $G = \langle \xi(\alpha), \xi(\beta) : \xi(\alpha)^5 = \xi(\beta)^5 = 1, \xi(\beta)\xi(\alpha)\xi(\beta)^{-1} = \xi(\alpha)^3 \rangle$ (it provides a family in $\mathcal{M}_{11}$)

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  (it provides a family in $\mathcal{M}_{11}$)
- The quotient surface $S/G = T$ is a torus with

$$\pi_G : S \to S/G = T$$

- The structures of T are given by the modular space $\mathcal{M}_1$.
- Note that the possition of the branch point in T does not play any role: all the points are conformally equivalents. $M_{(g=1,1)} = M_{g=1}$ (Birman 1975).

Subcase 2.

Action of generators of Modular group
$M_{(g=1,1)} = \langle X, Y : X^2 = Y^3 = 1 \rangle$

	a	b	x
X	b	a^{-1}	$a^{-1}xa$
Y	b	$a^{-1}b$	$a^{-1}xa$
XY	a^{-1}	$b^{-1}a^{-1}$	$b^{-1}a^{-1}xab$

The stratum for this subcase

Let $\mathcal{O}$ be an orbit of $M_{(g=1,1)}$ on the $\mathrm{Aut}(G)$-classes $\widetilde{K_G^\circ(1:g^G)}$, and let $\{\mathcal{O}_{X,i}:i\}$, $\{\mathcal{O}_{Y,j}:j\}$, $\{\mathcal{O}_{XY,k}:k\}$ be the orbit decomposition of $\mathcal{O}$ with respect to the cyclic subgroups $\langle\beta_{X*}\rangle$, $\langle\beta_{Y*}\rangle$, $\langle\beta_{XY*}\rangle$.
The one dimensional stratum is contained in a Riemann surface $\mathcal{S}$, and there is a covering $\mathcal{S}\to\mathcal{M}_1=\mathcal{M}_{(g=1,1)}$ branched over the orbifold singular points x, y of $\mathcal{M}_1$.

Theorem

1. *The degree of the covering is the size of the orbit* $|\mathcal{O}| = m$.
2. *Let* $\mathcal{O} = \{o_1, ..., o_m : o_i \in \mathrm{Aut}(G)$*-classes of* $K_G^\circ(1 : g^G)\}$. *The monodromy of the covering* $\mathcal{S} \to \mathbb{C}$ *is*

$$\omega : \pi_1(\mathbb{C} - \{x, y\}) = \langle X, Y :\rangle \to \Sigma_{|\mathcal{O}|} = \mathcal{P}\{1, ..., m\}$$

defined by $\omega(W(X, Y))(i) = j$ *if for each* $(a, b, x) \in o_i$ *we have that* $W(X, Y)(a, b, x) \in o_j$.

Corollary

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The equisymmetrical one dimensional strata in the moduli space of Riemann surfaces are Belyi surfaces curves with punctures.

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  Example: Moduli space: sphere without three points. Hurwitz space: sphere without four points.

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  Example: Moduli space: sphere without three points. Hurwitz space: sphere without four points.
- **Question.** Which Belyi curves appear as one dimensional equisymmetric strata?

The End

Thanks

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