Group actions on pseudo-real surfaces

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Joint work with
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Introduction

If G is a group of automorphisms of a compact Riemann surface of genus $g > 1$, then:

- $|G| \leq 84(g-1)$ if G preserves orientation [Hurwitz 1893]
- $|G| \leq 168(g-1)$ if G has orientation-reversing elements.

The largest order of an orientation-preserving automorphism of a compact Riemann surface of genus $g > 1$ is at most $4g+2$ [Wiman 1895].

The largest group of orientation-preserving automorphisms of a compact Riemann surface of genus $g > 1$ is at least $8g+8$ [Accola-Maclachlan 1968/69].

Also these bounds are sharp for infinitely many g.

Pseudo-real surfaces

A Riemann surface S is pseudo-real if it has anti-conformal automorphisms, but no such automorphism of order 2.

In the moduli space of compact Riemann surfaces of given genus, pseudo-real surfaces represent the points that have real moduli but are not definable over the reals.

Pseudo-real surfaces can admit glide reflections.

All pseudo-real surfaces have genus at least 2 (since every Riemann surface of genus 0 or 1 that admits an orientation-reversing automorphism admits a reflection of order 2).

Not much more was known about them until about 10 years ago, and a lot more has been found out in the last 2.5 years.

Group actions on pseudo-real surfaces

A group G is the automorphism group of a pseudo-real surface if there exists an epimorphism $\varphi\colon \Gamma \to G$ where Γ is an NEC-group with signature of the form $(\gamma; -; [m_1, ..., m_r]; \{\})$ and this cannot be extended to an epimorphism $\theta\colon \Phi \to H$ where Φ is an NEC-group containing a copy of Γ as a proper subgroup of finite index, with the same kernel as φ.

Possibilities for Γ and φ can be found with the help of group theory and MAGMA, and possibilities for the extension can be tested/eliminated using the 'reflexible' analogues of Singerman's classification of finitely-maximal Fuchsian groups, obtained by Bujalance (1982) and Estévez & Izquierdo (2006).

Note: The order of any orientation-reversing automorphism must be divisible by 4, and hence $|G|$ must be divisible by 4.

Some known facts about pseudo-real surfaces

[Bujalance, Conder & Costa (TAMS, 2010)]

1) There exists a pseudo-real Riemann surface of genus g, for every $g > 1$.

2) If S is a pseudo-real Riemann surface of genus $g > 1$, then $|\text{Aut}(S)| \leq 12(g-1)$ and the signature is $(1; -; [2,3]; \{\})$.

3) There exist infinitely many $g > 1$ for which some pseudo-real Riemann surface S of genus g has maximum symmetry, namely with the bound $|\text{Aut}(S)| \leq 12(g-1)$ being attained.

More recent discoveries

Various questions were raised at a BIRS workshop at Banff in September 2017 about pseudo-real surfaces and their automorphism groups. In joint work with Emilio Bujalance and Javier Cirre (Madrid) and PhD student Stephen Lo (UoA), we have now answered most of them. Our first answer is a partial generalisation of the Accola-Maclachlan bound:

4) Let $M(g)$ be the order of the largest group of automorphisms of a pseudo-real surface of genus $g > 1$. Then

$M(g) \geq 2g$ if g is even, while $M(g) \geq 4(g-1)$ if g is odd.

These bounds are sharp. The latter is sharp for a very large (and likely infinite) number of odd values of g. The degree of sharpness of the former bound is yet to be determined.

The next one generalises Wiman's theorem on the maximum order of an automorphism of a compact Riemann surface.

5) Let $M_{\text{cyc}}(g)$ and $M^{\text{o}}_{\text{cyc}}(g)$ be the largest order of an automorphism of a pseudo-real surface of genus $g > 1$ reversing or preserving orientation, respectively. Then

$M_{\text{cyc}}(g) \geq 2g$ if g is even, while $M_c(g) \geq 2(g-1)$ if g is odd, and

$M^{\text{o}}_{\text{cyc}}(g) \geq g,\ g-1,\ g,\ g+1$ when $g \equiv 0, 1, 2, 3$ mod 4 resp'ly.

Moreover, these bounds are sharp for infinitely many g.

The bound for $M_{\text{cyc}}(g)$ is attained by an action of C_{2g} with signature $(1; -; [2, 2, g]; \{\})$ when g is even, and an action of $C_{2(g-1)}$ with signature $(2; -; [2, 2]; \{\})$ when g is odd.
The bounds for $M^{\text{o}}_{\text{cyc}}(g)$ are attained by the orientation-preserving subgroups of certain abelian group actions.

The next one generalises a theorem of Maclachlan (1965) on the maximum order of an abelian group of automorphisms of a compact Riemann surface.

6) Let $M_{\text{ab}}(g)$ and $M^{\text{o}}_{\text{ab}}(g)$ be the largest order of an abelian group of automorphisms of a pseudo-real surface of genus $g > 1$ acting with or without orientation-reversing elements, respectively. Then

$M_{\text{ab}}(g) \geq 2g$, $2g+6$, $2g$, $2g+2$ when $g \equiv 0, 1, 2, 3$ mod 4,

and

$M^{\text{o}}_{\text{ab}}(g) \geq g$ if g is even, while $M^{\text{o}}_{\text{ab}}(g) \geq g+1$ if g is odd.

Also these bounds are sharp for a very large and possibly infinite set of values of g. (Proving the set is infinite is about as difficult as proving the twin primes conjecture.)

The symmetric genus of a group

The symmetric genus $\sigma(G)$ of a finite group G is the smallest genus of those compact Riemann surfaces on which G has a faithful action as a group of automorphisms, some of which might reverse orientation. Similarly, the strong symmetric genus $\sigma^{\text{O}}(G)$ of G is the smallest genus of the Riemann surfaces on which G has a faithful action preserving orientation.

By Hurwitz's theorem, $\sigma^{\text{O}}(G) \geq 1 + |G|/84$ when $\sigma^{\text{O}}(G) > 1$; and similarly $\sigma(G) \geq 1 + |G|/168$ when $\sigma(G) > 1$.

May & Zimmerman (2003) showed that every integer $g \geq 0$ is $\sigma^{\text{O}}(G)$ for some G. What about σ? Conder & Tucker (2011) proved that the range of σ covers over 88% of all non-negative integers, and by recent work of Stephen Lo, all but 5 integers in $[0 \ldots 1000]$ are known to be values of σ.

The pseudo-real genus of a group

The pseudo-real genus $\psi(G)$ of a finite group G is the smallest genus of those pseudo-real surfaces on which G acts faithfully as a group of automorphisms, some of which might reverse orientation.

Similarly, the strong pseudo-real genus $\psi^*(G)$ of G is the smallest genus of those pseudo-real surfaces on which G acts faithfully as a group of automorphisms, some of which do reverse orientation (when such a surface exists).

Question: What are the ranges of the functions ψ and ψ^*?

Some observations about ψ and ψ^*

a) $\psi^*(C_{4n}) = 2n$ for all $n > 0$, via an action with signature $(1; -; [2, 2, 2n]; \{-\})$, after smaller genera are eliminated;

b) $\psi^*(C_{4n} \rtimes_{2n-1} C_2) = 2n+1$ for all $n > 0$, via an action with signature $(1; -; [2, 2, 2]; \{-\})$, after smaller genera are eliminated;

c) If $G = C_{4n}$ or $C_{4n} \rtimes_{2n-1} C_2$, then $\psi(G) = \psi^*(G)$, with the proof of this taking some work.

Consequence: Every integer $g > 1$ occurs as both $\psi(G)$ and $\psi^*(G)$ for some G.

Final remarks

Clearly $\psi(G) \leq \psi^*(G)$ for every finite group G for which $\psi^*(G)$ is defined. Can it happen that $\psi(G) < \psi^*(G)$?

Yes! The smallest example is the 171st group of order 64 in the 'small groups' database (available in Magma).

For this group, $\psi^*(G) = 25$, via an action on such a surface with signature $(1, -; [2, 2, 4]; \{-\})$. On the other hand, G is isomorphic to a subgroup of index 2 in the 87th group of order 128, for which ψ^* takes value 17, via an action with signature $(1, -; [2, 4]; \{-\})$. Thus $\psi(G) = 17 < 25 = \psi^*(G)$.

Open question: For how many even values of g does the largest group of automorphisms of a pseudo-real surface of genus g have order $2g$?

Thank You!

Abstract

Title: Group actions on pseudo-real surfaces

Speaker: Marston Conder (University of Auckland)

Joint work with: Emilio Bujalance, Javier Cirre & Stephen Lo

Abstract:

A Riemann surface is *pseudo-real* if it admits anti-conformal automorphisms, but no anti-conformal automorphism of order 2. In the moduli space of compact Riemann surfaces

of given genus, pseudo-real surfaces represent the points that have real moduli, but are not definable over the reals. In this talk I'll describe a number of recent discoveries about pseudo-real surfaces, including analogues of theorems of Wiman and Maclachlan on the maximum order of a cyclic or abelian group of automorphisms. Also I'll discuss the *pseudo-real genus* $\psi(G)$ and the *strong pseudo-real genus* $\psi^*(G)$ of a finite group G. The former is the smallest genus of those pseudo-real surfaces on which G acts faithfully as a group of automorphisms, while the latter is the same but under the assumption that half the elements of G reverse orientation ... when such a surface exists. In particular, I'll show that for every $g > 1$ there is some G with $\psi(G) = \psi^*(G) = g$, so that the range of each of the functions ψ and ψ^* is complete. Finally, I'll give an example of a group G for which $\psi^*(G)$ is defined but $\psi(G) < \psi^*(G)$.