Geometric Realizations of Quasiplatonic Cyclic 8-gonal Surfaces

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Joint with Dami Lee
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Joint Work

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Introduction

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A surface is **polyhedral** if it is formed out of Euclidean polygons that meet only along edges. A polyhedral surface is **triply periodic** if the surface repeats along three independent directions via translation in $\mathbb{R}^3$. The above question is now refined as:

Question
Determine the conditions for which X is the quotient of a fundamental piece of a triply periodic polyhedral surface in $\mathbb{R}^3$.

Motivations

- Triply periodic polyhedral surfaces are the polyhedral analogues of minimal surfaces.
- Classifying the Riemann surfaces that are triply periodic helps determine when a Riemann surface can be embedded in the three-torus $\mathbb{R}^3/\Lambda$ for a lattice Λ.
- Plane sections of triply periodic surfaces are related to the theory of quasiperiodic functions in the plane and electron transport, in particular, particle diffusion in magnetic fields (De Leo, Maltsev 2018).

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4. Translation along $\langle 1,1,0\rangle, \langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$ directions generates the Octa-4 surface. This is a **fundamental piece** of the (infinite) Octa-4 triply periodic polyhedral surface.

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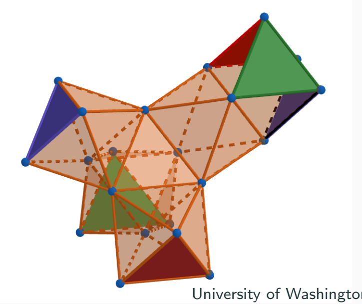
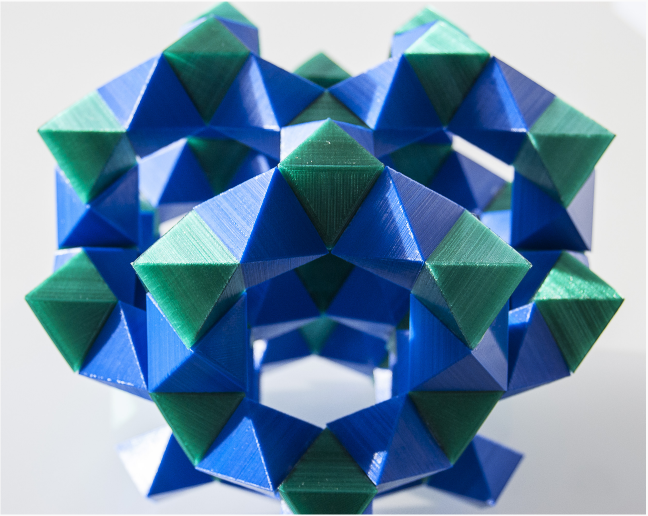
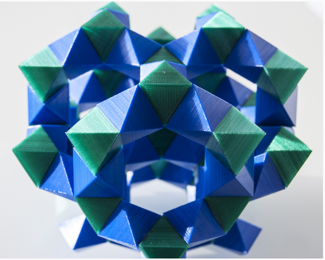
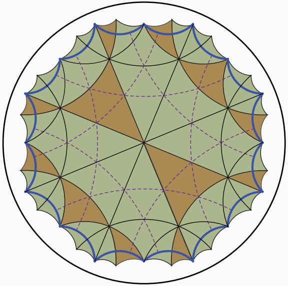


3D Printed



Theorem (Lee 2017)
The compact Riemann surface obtained as the quotient of the above fundamental piece by translation is conformally equivalent to Fermat quartic $\{(x, y) \in \mathbb{C}^2 : x^4 + y^4 = 1\}$ *of genus three.*





Centers = green octahedra,

Handles = blue octahedra.

Brown triangles come from center octahedra, lighter triangles come from handle octahedra.

The Correspondence

- Given a fundamental piece K of a triply periodic polyhedral surface Π, how can we associate to K a connected region of hyperbolic polygons with edge identifications, tessellating the hyperbolic disc, whose polygons are in bijection with the Euclidean polygons of Π?

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- Given a fundamental piece K of a triply periodic polyhedral surface Π, how can we associate to K a connected region of hyperbolic polygons with edge identifications, tessellating the hyperbolic disc, whose polygons are in bijection with the Euclidean polygons of Π?
- Conversely, can the hyperbolic tessellation of a given compact Riemann surface yield a triply periodic surface?

Euler Characteristic

Let

- P: closed polyhedral surface formed out of regular Euclidean p-gons, q meeting at each vertex.
- V, E, F: number of vertices, edges, and faces, respectively, of P.

Since $V = pF/q$ and $E = pF/2$, then $V - E + F = 2 - 2g$ is rewritten in terms of F as

$$F_g(p, q) := F = \frac{4q(g-1)}{(p-2)(q-2)-4}.$$

Number of Faces

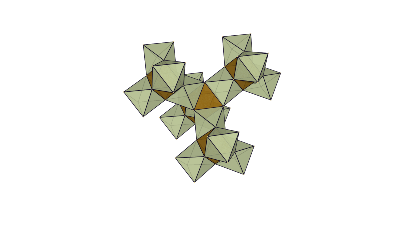
- Given a (infinite) triply periodic polyhedral surface, a **fundamental piece** is a finite, connected subcollection of polygons that spans the entire surface under translation in three independent directions.
- A fundamental piece K of a (regular) triply periodic polyhedral surface has an associated lattice Λ in $\mathbb{R}^3$ invariant under translation. The closed polyhedral quotient surface $P = K/\Lambda$ with Shlafli symbol $\{p, q\}$ is then formed from identifications via translation.
- $F_g(p, q) = \dfrac{4q(g-1)}{(p-2)(q-2)-4}$ relates the number of faces of P, the type $\{p, q\}$ and the genus g of P.

Platonic Realizations

$X \cong \mathbb{H}/\Gamma$ is a **Platonic surface** if $\Gamma \lhd \Delta(2, p, q)$ for some hyperbolic triangle group $\Delta(2, p, q)$. These are the known triply periodic polyhedral representations P of the Platonic surfaces for genus $g = 3, 4$ with the number of faces $F_g(p, q)$ of the corresponding fundamental piece of P.

$\underline{g = 3}$:

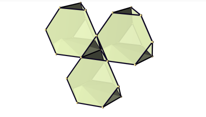
- $\Delta(2, 3, 8), F_3(3, 8) = 32$ triangles: Octa-4



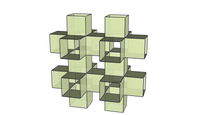
Platonic Realizations

$\underline{g = 3}$ (cont.):

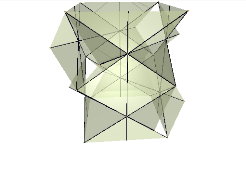
- $\Delta(2,6,6)$, $F_4(6,6) = 4$ regular hexagons: Mutetrahedron



- $\Delta(2,4,6)$, $F_3(4,6) = 12$ squares: Cube-6



- $\Delta(2,4,8)$, 32 isosceles triangles or 16 rhombi: Schwarz-CLP surface (research in progress)

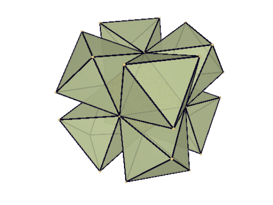


- Still unknown for $g = 3$ (to our knowledge): $\Delta(2,3,7)$ (Klein quartic), $\Delta(2,3,12)$, $\Delta(2,4,12)$, $\Delta(2,7,14)$, $\Delta(2,8,8)$, $\Delta(2,12,12)$.

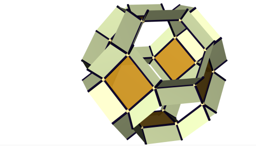
Platonic Realizations

$g = 4$:

- $\Delta(2,3,12)$, $F_4(3,8) = 24$ triangles: Octa-8



- $\Delta(2,4,5)$, $F_4(4,5) = 30$ squares: Truncated Octa-8
- $\Delta(2,5,5)$, $F_4(5,5) = 12$ pentagons: also Truncated Octa-8



Still unknown for $g = 4$ (to our knowledge): $\Delta(2,9,18)$, $\Delta(2,4,16)$, $\Delta(2,4,10)$, $\Delta(2,6,6)$, $\Delta(2,4,6)$, $\Delta(2,16,16)*$, $\Delta(2,10,10)*$, $\Delta(2,6,12)*$.
$*$Non-maximal in genus four and may have an overgroup with a triply periodic representation.

The Quasiplatonic C_8 Surfaces and New Triply Periodic Constructions

- A group action $G \hookrightarrow \mathrm{Homeo}^+(X)$ of a finite group G on a surface X is **quasiplatonic** if X/G is genus zero and three branching values. Up to topological equivalence, there are three quasiplatonic C_8-actions on surfaces of genus two or greater. These three actions have generating vectors $[1,3,4], [1,2,5]$ and $[1,1,6]$ acting on conformally distinct Riemann surfaces of genus 2 (Bolza surface), 3 (Fermat quartic) and 3, respectively.
- We study the extent to which these surfaces have triply periodic polyehdral representations in $\mathbb{R}^3$.

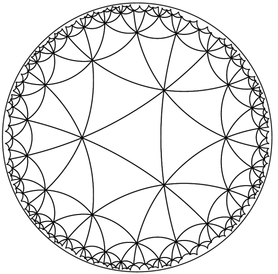
Known and In-Progress

- We already know Fermat quartic, with the $[1, 2, 5]$ generating vector for C_8, has a triply periodic polyhedral representation (Lee 2017).
- We are currently verifying that the $[1, 1, 6]$ surface of genus three can be represented by the Schwarz-CLP surface.

Making a Triply Periodic Surface from Scratch

Question
How can one build a triply periodic surface from a hyperbolic tiling coming from a Riemann surface?

We associate a net of regular Euclidean polygons to a hyperbolic tiling by finding simple closed paths passing through the hyperbolic polygons. We are currently exploring this idea based on triangular tilings.

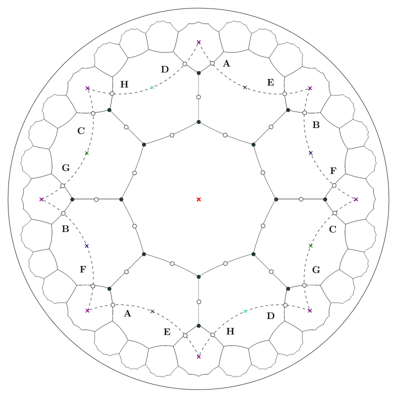


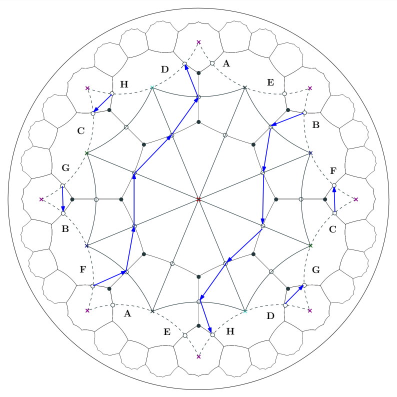
$$\Rightarrow \left\{ \begin{array}{c} \text{Net of Euclidean} \\ \text{Equilateral Triangles} \end{array} \right\}$$

Driving Directions

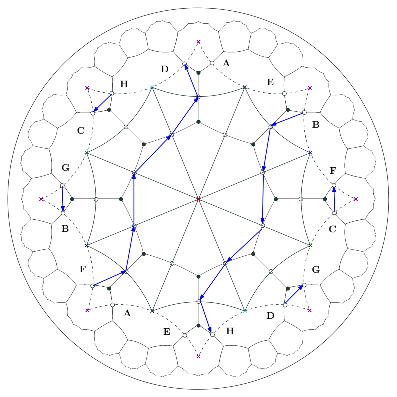
Starting from the centers of certain edges, performing the directions "Left-Left-Right-Right" results in a simple closed path passing through 16 triangles. For each of the three quasiplatonic C_8-actions and their respective edge identifications in the hyperbolic plane, we can achieve nets of Euclidean triangles.

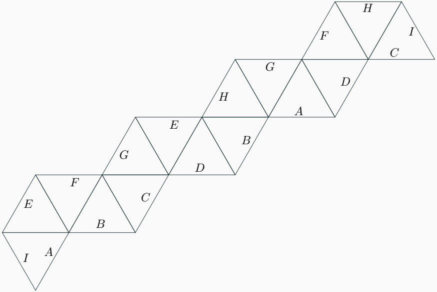
$[1, 3, 4]$ **Generating Vector (Bolza Surface, Genus Two)**





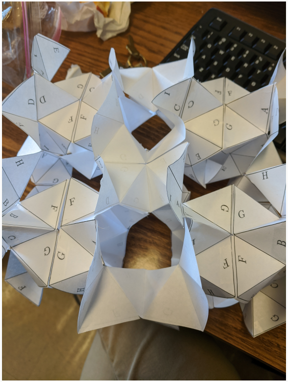
$[1,3,4]$: Directions to Net

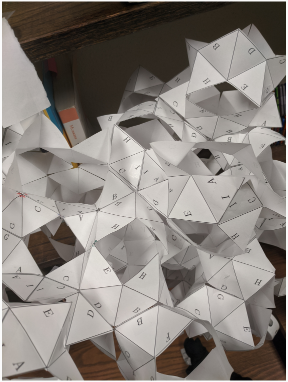




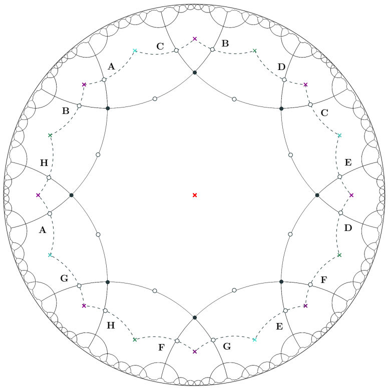
Though a surface of genus two cannot yield a triply periodic surface, it may have a covering which is triply periodic. This can be achieved by physically gluing many copies of the associated net.

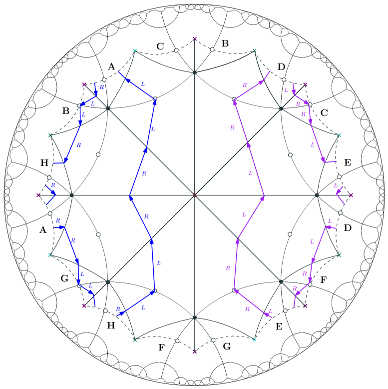
One attempt (using 256 triangles on the left):



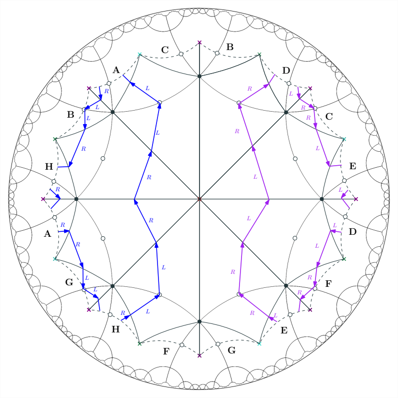


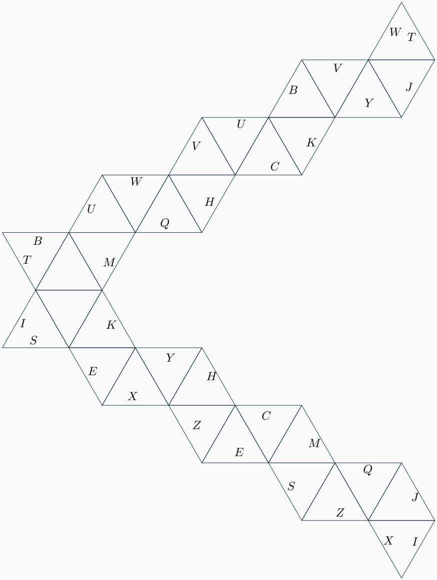
$[1, 1, 6]$ Generating Vector (Surface of Genus Three)





$[1, 1, 6]$: Directions to Net





$[1,3,4]$: Net to Triply Periodic Covering?

It is unclear whether we can make a triply periodic polyhedral surface from only one $[1,1,6]$ net, but it may be possible to make a (non-regular) triply periodic surface.

One attempt (using 384 triangles!):

It seems that most of the Platonic surfaces of type $\{p, q\}$ of genus $g \geq 3$ have triply periodic representations, as long as the number of faces $F_g(p, q)$ is small enough. We conjecture:

Conjecture
Let X be a compact Riemann surface of genus three or greater that is Platonic with respect to the $\{p, q\}$ tiling, excluding the case $\{3, 7\}$. If X admits either a C_q-action with signature $(0; n_1, \ldots, n_p)$ or a C_p-action with signature $(0; m_1, \ldots, m_q)$, and $F_g(p, q) \geq q/2$, then X can be geometrically realized as a quotient of a fundamental piece of a triply periodic polyhedral surface of type $\{p, q\}$.

Some Questions You May Have...

Can the polyhedral surfaces be formed out of more than one size of Euclidean polygon, e.g., out of triangles and squares?

Yes!

What happens when you consider larger fundamental pieces to a triply periodic surface?

Larger fundamental pieces yield coverings of the smallest fundamental piece.

References

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- Marston Conder's webpage: https://www.math.auckland.ac.nz/ conder/TriangleGroupQuotients101.txt

Thank you!

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