Classifying Pairs of Fuchsian Groups of Finite Type

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Outline

Pairs of Fuchsian Groups, Why?

- consider finite index pairs $\Gamma \subseteq \Delta$ of finite area Fuchsian groups, *FG*-pairs for short
- goal: classify *FG*-pairs up to various types of equivalence
- first consider three motivations

Motivation 1 - extending actions -1

- suppose X is a closed surface with group of automorphisms H
- when is H the full automorphism group
- suppose X admits an overgroup $G \supseteq H$ of automorphisms
- there are Fuchsian groups $\Gamma \leq \Delta$ and $\Pi \trianglelefteq \Delta$ with $X \simeq \mathbb{H}/\Pi$
- and Γ and Δ uniformize the action of H and G respectively

Motivation 1 - extending actions - 2

- uniformization of H-action determined by $\eta : \Gamma \to H$
- select Δ from the classification (if it exists, moduli problem)
- H-action on X extends to $G \simeq \Delta/\Pi$ if there is an extension $\eta : \Delta \to G$
- whether η extends can be determined by knowing the inclusion map $\Gamma \to \Delta$ in algebraic form

Motivation 1 - finitely maximal - 1

- Γ is finitely maximal if there is no overgroup of finite index
- action can extend only in case Γ is not finitely maximal
- Greenberg and later Singerman worked out a finite list of Fuchsian groups that are never finitely maximal
- action extension is now just an algebraic problem, no moduli constraints
- this case by worked out by Bujalance, Conder and Cirre

Motivation 2 - divisible tilings 1

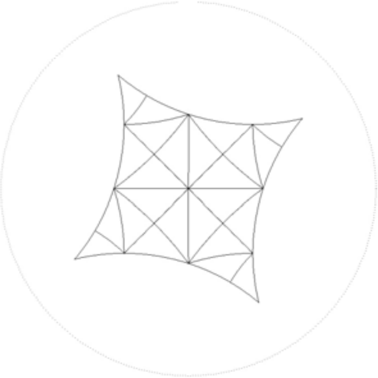
- suppose P is a convex hyperbolic polygon whose angles have the form π/n for $n \in \{2, 3, 4, \ldots, \infty\}$
- called a kaleidoscopic polygon, see slide after next
- P generates a tiling by repeated reflection in sides
- reflections in sides of polygon generate a crystallographic group
- define Γ to be the subgroup of orientation preserving transformations

Motivation 2 - divisible tilings 2

- suppose $Q \subset P$ is a kaleidoscopic polygon whose tiling refines first tiling, namely P is tiled by repeated reflections of Q
- let Δ be the Fuchsian group determined by Q
- then $\Gamma \subset \Delta$ is a FG-pair where the index is the number of Q-polygons tiling a P-polygon

Motivation 2 - divisible tilings 3

- P = large polygon, angles = $(\pi/5, \pi/5, \pi/5, \pi/5)$
- Q = small polygon, angles = $(\pi/2, \pi/4, \pi/5)$



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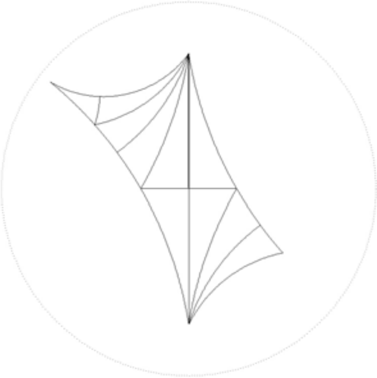
Motivation 2 - divisible tilings 4

- P = large polygon, angles = $(\pi/3, \pi/4, \pi/5, \pi/20)$
- Q = small polygon, angles = $(\pi/2, \pi/3, \pi/20)$



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Motivation 2 - divisible tilings 5

- in the last frame observe that there are “freely moveable” vertices
- we get an entire family of polygon pairs $Q \subset P$ where P has angles $(\pi/3, \pi/4d, \pi/5d, \pi/20d)$, Q has angles $(\pi/2, \pi/3, \pi/20d)$, and $d \geq 1$ is an integer parameter
- the limiting polygon as $d \to \infty$ has cusps on the boundary of $\mathbb{H}$
- the entire family is determined by the limiting cusp case
- the algebraic structure of Δ/Γ is independent of the order at the free vertices.

Motivation 3 - stratification of moduli space

- moduli space $\mathfrak{M}_\sigma$ is the space of conformal equivalence classes of surfaces of genus σ
- $\mathfrak{M}_\sigma$ may be stratified into a finite disjoint union of locally closed smooth subvarieties, consisting of surfaces of similar symmetry type, called equisymmetric strata
- all surfaces in a given stratum have automorphism group isomorphic to a fixed group H (all conjugate in the mapping class group)

Motivation 3 - stratification of moduli space 3

- if Σ is a stratum its closure is a union of strata of lower dimension(adjunction) corresponding to overgroups $G \supseteq H$
- $G \supseteq H$ determines an FG-pair $\Gamma \subset \Delta$
- classification of the FG-pairs helps determine the equisymmetric structure of $\mathfrak{M}_\sigma$ (adjunction relation)

Fuchsian Groups - presentation

- Γ has a presentation

$$\text{generators}: \{\alpha_i, \beta_i, \gamma_j, \delta_k, 1 \leq i \leq \sigma, 1 \leq j \leq s, 1 \leq k \leq p\}$$

$$\text{relations}: \prod_{i=1}^{\sigma}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j \prod_{k=1}^{p} \delta_k = \gamma_1^{m_1} = \cdots = \gamma_s^{m_s} = 1$$

- the signature of Γ is

$$\mathcal{S}(\Gamma) = (\sigma : m_1, \ldots, m_s, m_{s+1}, \ldots, m_{s+p})$$

with $m_{s+j} = \infty, j = 1, \ldots, p$

Fuchsian Groups - invariants

- important invariants of a Fuchsian group
- the *genus of* Γ: $\sigma(\Gamma) = \sigma$ is the genus of $S = \overline{\mathbb{H}/\Gamma}$
- area of a fundamental region: $A(\Gamma) = 2\pi\mu(\Gamma)$ where:

$$\mu(\Gamma) = 2(\sigma - 1) + \sum_{j=1}^{s+p}(1 - \frac{1}{m_j}).$$

- *Teichmüller dimension* $d(\Gamma)$ of Γ: the dimension of the Teichmüller space of Fuchsian groups with signature $\mathcal{S}(\Gamma)$ given by
$$d(\Gamma) = 3(\sigma - 1) + s + p.$$

Fuchsian Group Pairs - codimension

- for finite index *FG*-pair $\Gamma \subseteq \Delta$ we call the quantity $d(\Gamma, \Delta) = d(\Gamma) - d(\Delta)$ the *Teichmüller codimension* of (Γ, Δ)
- in Motivation 1: Singerman's list is the list of codimension 0 pairs.
- in Motivation 2: the tiling of an *m*-gon by an *n*-gon determines a codimension $m - n$ pair
- Motivation 3: If Σ_2 lies in the closure of Σ_1 then a codimension $dim(\Sigma_1) - dim(\Sigma_2)$ pair is determined

Fuchsian Group Pairs - monodromy

- the pair $\Gamma \subseteq \Delta$ of index n determines a permutation representation $q : \Delta \to \Sigma_n$ where n is the index of Γ in Δ
- the representation may be captured geometrically from the monodromy of the branched covering $S \to T$ $S = \overline{\mathbb{H}/\Gamma}$, $T = \overline{\mathbb{H}/\Delta}$
- a generating set for Δ determines a sequence

$$\{q(\alpha_i), q(\beta_i), q(\gamma_j), q(\delta_k), 1 \leq i \leq \sigma, 1 \leq j \leq s, 1 \leq k \leq p\}$$

  of elements of Σ_n satisfying certain properties
- the sequence above is called a monodromy vector
- monodromy vector structure determined entirely by the signatures of Γ and Δ
- $M(\Delta, \Gamma) = q(\Delta)$ is called the monodromy group of the pair

Overview

- classify by codimension
- constrained and tight pairs
- finiteness of classification in each codimension
- numerical projections
- primitive pairs , $M(\Delta, \Gamma)$ is a primitive permutation group
- Towers of Fuchsian groups

Steps of Classification - 1

- determine all signature pairs for a fixed codimension
- can be done by computer search resulting in
- finitely many exceptional cases
- finitely many families

Overview

- classify by codimension
- for each codimension the dimension $d(\Gamma)$ and hence $\sigma(\Gamma)$ is bounded
- only classify primitive pairs , i.e., $M(\Delta, \Gamma)$ is a primitive permutation group
- classify by algebraic and then by conformal equivalence
- each algebraic equivalence class determines a moduli space of conformally inequivalent pairs
- each moduli space “looks like” the moduli space of the containing Fuchsian group (finite cover)

Steps of Classification - 2

- for each candidate signature pair, compute all the compatible monodromy vectors up to algebraic equivalence
- use computer calculation and classification of primitive permutation groups (use Magma or GAP)
- using the Riemann existence theorem, each monodromy vector determines an *FG*-pair
- in turn a moduli space of pairs is determined.

Steps of Classification - 3

- for each monodromy vector a branched cover $S \to T$ may be constructed
- the inclusion $\Gamma \to \Delta$, written in terms of a canonical generating set, may be constructed from the branched cover

General results

- for each codimension there is a finite classification as follows
- finitely many exceptional pairs without punctures
- finitely many families, each parameterized by integers
- each family is derived form a single pair with punctures

By codimension

- Codimension 0 - Singerman's list - inclusions have been calculated - Bujalance, Conder and Cirre
- Codimension 1 - all cases coming from polygonal pairs determined by previous efforts of author and students -inclusions not yet calculated
- codimension 2 - most signature pairs calculated, this case appears to be computable
- higher codimension cases may become very computationally intensive

Future Work

- finish extendibility algorithm, i.e., automatic computation of algebraic form of inclusion $\Gamma \to \Delta$ in algebraic form from the monodromy vector
- completely work out structure of moduli space (adjunction relation) for low genus, say genus 3 and 4
- Teichmüller curves in Teichmüller space - dimension 1 strata and the zero dimensional strata lying on them (triangle groups in quadrilateral groups and one other case)

References

- *On extendability of group actions on compact Riemann surfaces*, Bujalance, Conder and Cirre, Trans. Amer. Math. Soc. 355 (2003), 1537-1557.
- *Divisible Tilings in the Hyperbolic Plane* (with Dawn M. Haney, Lori T. McKeough, Brandy M. Smith), New York Journal of Mathematics 6 (2000), 237-283.

- slides will be posted here *https://www.rose-hulman.edu/mapub/autosurf/Tuscson2007.html*