The Lattice Structure of the Potential Signature Space

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2. Naive Statements of Goals and Results
3. Determining Group Actions
4. Potential and Actual Signatures
5. Consequences

Introduction

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The following are the two conditions necessary for the existence of a group action on a surface of genus σ:

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Potential Signatures

We can now formally define the objects that satisfy the arithmetic conditions for the existence of a group action:

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$\mathcal{P}_\sigma$ is the set of tuples, for which, given any such tuple $(h; m_1, \ldots, m_r)$ there exists an integer $N > 0$ such that:

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Example: Potential Signatures in Genus 2

Example

$$\mathcal{P}_2 = \left\{ \begin{array}{cccc} (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}) & (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}) & (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{3}) & (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{4}) \\ (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{3}, \mathbf{3}) & (\mathbf{0}; \mathbf{2}, \mathbf{2}, \mathbf{4}, \mathbf{4}) & (\mathbf{0}; \mathbf{2}, \mathbf{3}, \mathbf{8}) & (\mathbf{0}; \mathbf{2}, \mathbf{4}, \mathbf{6}) \\ (\mathbf{0}; \mathbf{2}, \mathbf{4}, \mathbf{8}) & (\mathbf{0}; \mathbf{2}, \mathbf{5}, \mathbf{10}) & (\mathbf{0}; \mathbf{2}, \mathbf{6}, \mathbf{6}) & (\mathbf{0}; \mathbf{2}, \mathbf{8}, \mathbf{8}) \\ (\mathbf{0}; \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}) & (\mathbf{0}; \mathbf{3}, \mathbf{3}, \mathbf{4}) & (\mathbf{0}; \mathbf{3}, \mathbf{4}, \mathbf{4}) & (\mathbf{0}; \mathbf{3}, \mathbf{6}, \mathbf{6}) \\ (\mathbf{0}; \mathbf{4}, \mathbf{4}, \mathbf{4}) & (\mathbf{0}; \mathbf{5}, \mathbf{5}, \mathbf{5}) & (\mathbf{1}; \mathbf{2}, \mathbf{2}) & (\mathbf{2}; -) \\ (1; 3) & (0; 3, 3, 6) & (0; 3, 3, 5) & (0; 2, 3, 18) \\ (0; 2, 5, 5) & (0; 2, 3, 12) & (0; 3, 3, 9) & (0; 2, 3, 10) \\ (0; 2, 3, 9) & (0; 2, 4, 5) & (0; 2, 3, 7) & (0; 2, 2, 2, 6) \\ (1; 2) & & & \end{array} \right\}$$

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$\mathcal{A}_\sigma$ is the set of signatures for which there exists an action of some finite group G on a surface of genus σ with that signature. We call these *actual signatures*.

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Formalizing our Goals

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Can we describe the relationship between the potential signature spaces $\mathcal{P}_\sigma$ as we vary σ? What does it tell us about the actual signature spaces $\mathcal{A}_\sigma$ as we vary σ?

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Our First Main Result

Theorem

The set $\{\mathcal{P}_\sigma\}_{\sigma \geq 2}$ forms a lattice with ordering determined by the divisibility of $\sigma - 1$. Specifically,

- *(The meet) If $\gcd((\sigma - 1), (\sigma' - 1)) = \Sigma - 1$, then $\mathcal{P}_\sigma \cap \mathcal{P}_{\sigma'} = \mathcal{P}_\Sigma$.*
- *(The join) If $\mathrm{lcm}((\sigma - 1), (\sigma' - 1)) = \Sigma - 1$, then $\mathcal{P}_\sigma \cup \mathcal{P}_{\sigma'} = \mathcal{P}_\Sigma$.*

A Picture Paints a Thousand Words

$$
\begin{tikzcd}
\mathcal{P}_{31} \arrow[d] \arrow[dr] & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathcal{P}_{7} \arrow[d] \arrow[dr] & \mathcal{P}_{16} \arrow[d] \arrow[dr] & \cdots & \cdots & \mathcal{P}_{pq+1} \arrow[d] \arrow[drr] & \cdots & \cdots & \cdots \\
\mathcal{P}_{3} \arrow[drr] & \mathcal{P}_{4} \arrow[dr] & \mathcal{P}_{6} \arrow[d] & \cdots & \mathcal{P}_{p+1} \arrow[dll] & \cdots & \mathcal{P}_{q+1} \arrow[dllll] & \cdots \\
 & & \mathcal{P}_{2} & & & & &
\end{tikzcd}
$$

Figure: A Partial Representation of the Potential Signature Space

First Consequence: Omnipersistent Potential Signatures

Only a small number of signatures *could* appear in every possible genus:

Theorem

The omnipersistent potential signatures are:

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- This partially explains why we so regularly see certain signatures in many genera e.g. $(0;2,3,7)$ keeps showing up!

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$$\mathcal{P}_2 = \left\{ \begin{array}{cccc} (0;2,2,2,2,2) & (0;2,2,2,2,2,2) & (0;2,2,2,3) & (0;2,2,2,4) \\ (0;2,2,3,3) & (0;2,2,4,4) & (0;2,3,8) & (0;2,4,6) \\ (0;2,4,8) & (0;2,5,10) & (0;2,6,6) & (0;2,8,8) \\ (0;3,3,3,3) & (0;3,3,4) & (0;3,4,4) & (0;3,6,6) \\ (0;4,4,4) & (0;5,5,5) & (1;2,2) & (2;-) \\ (1;3) & (0;3,3,6) & (0;3,3,5) & (0;2,3,18) \\ (0;2,5,5) & (0;2,3,12) & (0;3,3,9) & (0;2,3,10) \\ (0;2,3,9) & (0;2,4,5) & (0;2,3,7) & (0;2,2,2,6) \\ (1;2) & & & \end{array} \right\}$$

• This partially explains why we so regularly see certain signatures in many genera e.g. $(0;2,3,7)$ keeps showing up!

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