The Lattice Structure of the Potential Signature Space

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Overview

1. Introduction
2. Naive Statements of Goals and Results
3. Determining Group Actions
4. Potential and Actual Signatures
5. Consequences
The underlying question behind much of my work is the following:

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*Can we describe the possible finite conformal group actions on compact Riemann surfaces of genus 2 and higher?*
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To prove the existence of a group action on a surface $X$ of genus $\sigma$, there are two sets of conditions that need verifying:

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Our approach to the problem

In light of these difficulties, we have taken the following approach:

1. Consider the arithmetic conditions first (the “easy” step). Specifically:
   - What do the arithmetic conditions tell us in a specific genus $\sigma$?
   - How do these compare between different genera?
2. Consider group theoretic conditions second (the “difficult” step)
   - What happens in a particular genus when we impose this additional condition?
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• The focus of our initial research is this first step.
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Suppose that a finite group $G$ acts on a surface $X$ of genus $\sigma$.

**Definition**

We say that $G$ has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

1. The quotient space $X/G$ has genus $h$.
2. The quotient map $\pi: X \to X/G$ is branched over $r$ points with branching orders $m_1, \ldots, m_r$. 
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Lattice Structure

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Suppose that $G$ is a finite group and $S = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $V = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of $G$ is an $S$-generating vector for $G$ if the following hold:
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**Theorem**

A group $G$ acts on a surface $X$ of genus $\sigma$ with signature $S = (h; m_1, \ldots, m_r)$ if and only if the following hold:

1. The Riemann Hurwitz formula holds:
   \[ \sigma - 1 = |G| (h - 1) + 2 \sum_{i=1}^{r} (1 - 1/m_i), \]

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Potential Signatures

We can now formally define the objects that satisfy the arithmetic conditions for the existence of a group action:

**Definition**

$P_\sigma$ is the set of tuples, for which, given any such tuple $(h; m_1, \ldots, m_r)$ there exists an integer $N > 0$ such that:

1. Each $m_i | N$
2. $\sigma^{-1} = N (h - 1) + \sum_{i=1}^{r} (1 - 1/m_i)$

We call these potential signatures.

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Example: Potential Signatures in Genus 2

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Actual Signatures

**Definition**

\( \mathcal{A}_\sigma \) is the set of signatures for which there exists an action of some finite group \( G \) on a surface of genus \( \sigma \) with that signature. We call these *actual* signatures.

---

\[ \mathcal{A}_\sigma \subseteq \mathcal{P}_\sigma \text{ for every } \sigma. \]
Actual Signatures

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Example

\[
\mathcal{A}_2 = \left\{ (0; 2, 2, 2, 2, 2), (0; 2, 2, 2, 2, 2), (0; 2, 2, 2, 3), (0; 2, 2, 2, 4), (0; 2, 2, 3, 3), (0; 2, 2, 4, 4), (0; 2, 3, 8), (0; 2, 4, 6), (0; 2, 4, 8), (0; 2, 5, 10), (0; 2, 6, 6), (0; 2, 8, 8), (0; 3, 3, 3, 3), (0; 3, 3, 4), (0; 3, 4, 4), (0; 3, 6, 6), (0; 4, 4, 4), (0; 5, 5, 5), (1; 2, 2), (2; -) \right\}
\]
**Actual Signatures**

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### Example

\[
\mathcal{A}_2 = \begin{cases} 
(0; 2, 2, 2, 2, 2) & (0; 2, 2, 2, 2, 2) & (0; 2, 2, 2, 3) & (0; 2, 2, 2, 4) \\
(0; 2, 2, 3, 3) & (0; 2, 2, 4, 4) & (0; 2, 3, 8) & (0; 2, 4, 6) \\
(0; 2, 4, 8) & (0; 2, 5, 10) & (0; 2, 6, 6) & (0; 2, 8, 8) \\
(0; 3, 3, 3, 3) & (0; 3, 3, 4) & (0; 3, 4, 4) & (0; 3, 6, 6) \\
(0; 4, 4, 4) & (0; 5, 5, 5) & (1; 2, 2) & (2; -) 
\end{cases}
\]

- Note: \( \mathcal{A}_\sigma \subseteq \mathcal{P}_\sigma \) for every \( \sigma \).
**Actual Signatures**

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$A_\sigma$ is the set of signatures for which there exists an action of some finite group $G$ on a surface of genus $\sigma$ with that signature. We call these *actual signatures*.

**Example**

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A_2 = \left\{ \begin{array}{cccc}
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(0; 2, 2, 3, 3) & (0; 2, 2, 4, 4) & (0; 2, 3, 8) & (0; 2, 4, 6) \\
(0; 2, 4, 8) & (0; 2, 5, 10) & (0; 2, 6, 6) & (0; 2, 8, 8) \\
(0; 3, 3, 3, 3) & (0; 3, 3, 4) & (0; 3, 4, 4) & (0; 3, 6, 6) \\
(0; 4, 4, 4) & (0; 5, 5, 5) & (1; 2, 2) & (2; -) \\
\end{array} \right\}
$$

- Note: $A_\sigma \subseteq P_\sigma$ for every $\sigma$. 
Formalizing our Goals

Goal

Can we describe the relationship between the potential signature spaces $\mathcal{P}_\sigma$ as we vary $\sigma$? What does it tell us about the actual signature spaces $\mathcal{A}_\sigma$ as we vary $\sigma$?
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Our First Main Result

Theorem

The set \( \{ P_\sigma \}_{\sigma \geq 2} \) forms a lattice with ordering determined by the divisibility of \( \sigma - 1 \). Specifically,

- (The meet) If \( \gcd((\sigma - 1), (\sigma' - 1)) = \Sigma - 1 \), then \( P_\sigma \cap P_{\sigma'} = P_\Sigma \).
- (The join) If \( \text{lcm}((\sigma - 1), (\sigma' - 1)) = \Sigma - 1 \), then \( P_\sigma \cup P_{\sigma'} = P_\Sigma \).
Figure: A Partial Representation of the Potential Signature Space
First Consequence: Omnipersistent Potential Signatures

Only a small number of signatures could appear in every possible genus:

Theorem

The omnipersistent potential signatures are:

\[ \mathcal{P}_2 = \left\{ \begin{array}{cccc}
(0; 2, 2, 2, 2, 2) & (0; 2, 2, 2, 2, 2) & (0; 2, 2, 2, 3) & (0; 2, 2, 2, 4) \\
(0; 2, 2, 3, 3) & (0; 2, 2, 4, 4) & (0; 2, 3, 8) & (0; 2, 4, 6) \\
(0; 2, 4, 8) & (0; 2, 5, 10) & (0; 2, 6, 6) & (0; 2, 8, 8) \\
(0; 3, 3, 3, 3) & (0; 3, 3, 4) & (0; 3, 4, 4) & (0; 3, 6, 6) \\
(0; 4, 4, 4) & (0; 5, 5, 5) & (1; 2, 2) & (2; -) \\
(1; 3) & (0; 3, 3, 6) & (0; 3, 3, 5) & (0; 2, 3, 18) \\
(0; 2, 5, 5) & (0; 2, 3, 12) & (0; 3, 3, 9) & (0; 2, 3, 10) \\
(0; 2, 3, 9) & (0; 2, 4, 5) & (0; 2, 3, 7) & (0; 2, 2, 2, 6) \\
(1; 2) & 
\end{array} \right\} \]
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- This partially explains why we so regularly see certain signatures in many genera e.g. (0; 2, 3, 7) keeps showing up!
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Second Consequence: Numbers of Potential Signatures Grow with Divisibility

We have the following easy consequence given by the lattice structure:

**Theorem**

\[ \mathcal{P}_\sigma \subseteq \mathcal{P}_{\sigma'} \text{ if and only if } (\sigma - 1) | (\sigma' - 1). \]

One direction is given by the lattice theorem. For the other direction, simple application of the Riemann-Hurwitz formula shows that if \((\sigma - 1) \nmid (\sigma' - 1)\), then \((0, 2, 2g + 1, 4g + 2) \in \mathcal{P}_\sigma\) but \((0, 2, 2g + 1, 4g + 2) \not\in \mathcal{P}_{\sigma'}\).

This partially explains the growth rates in the number of group actions in different genera – when \(\sigma - 1\) has lots of divisors \(\mathcal{P}_\sigma\) is large!
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We know all omnipersistent potential signatures. This motivates:

**Question**

Are there any omnipersistent actual signatures?

**Theorem**

The omnipersistent signatures are \{ (2; 0), (1; 2^2), (0; 2^2, 2^2, 2^2) \}.

**Proof.**

\((x, e, x, e)\) is a \((2; -1)\)-generating vector for \(C_{\sigma - 1} = \langle x \rangle\).

\((x, e, y, y)\) is a \((1; 2^2)\) and \((y, y, xy, xy, y, y)\) is a \((0; 2^2, 2^2, 2^2, 2^2, 2^2)\)-generating vector for \(D_{\sigma - 1} = \langle x, y | x_{\sigma - 1}, y^2, yxyx \rangle\).

\((xy, xy, y, y)\) is a \((0; 2^2, 2^2, 2^2, 2^2, 2^2)\)-generating vector for \(D_2(\sigma - 1) = \langle x, y | x_{2(\sigma - 1)}, y^2, yxyx \rangle\).
Final Observation

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A. Wootton  (University of Portland)  Lattice Structure  April 13, 2018  18 / 18
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