Unfaithful Maps.

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For us, two group actions are equivalent if they are conjugate by a homeomorphism of the surface.

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Leads to Riemann-Hurwitz equation relating $\chi(S)$ to $\chi(S/G)$ with branching information, which dominates the study of which surfaces a given group can act on (e.g $|G| \leq 84(\gamma(S) - 1)$).

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This can be proved by realizing the orbifold S/G by a fundamental polygon in the hyperbolic plane.

This allows us to think of S as a Riemann surface with a complex structure preserved by G and give us the type of the surface, this time by the fundamental polygon.

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how to glue flags together along sides a, b, c is given by three involutary permutations x, y, z on set of flags satisfying $(xy)^2 = 1$.
Thus map is simply a transitive (since S connected) permutation group $Mon(M)$ with a marked generating set x, y, z of involutions with $(xy)^2 = 1$, where vertices, edges, and faces are cosets of $\langle x, z\rangle, \langle x, y\rangle, \langle y, z\rangle$.

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Thus finite group actions on closed surfaces same as study of centralizers of transitive permutation groups with marked generating set of three involutions...

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And can compose $*$ and P: 6 possibilities in all since $\langle x, y \rangle = C_2 \times C_2$ has 6 choices for the ordered pair playing the role of x, y.
Note that $Aut(M^*)$ and $Aut(M^P)$ are isomorphic to $Aut(M)$, since the permutation group $Mon(M)$ does not change, only the marked generating set changes by choices in $\langle x, y \rangle$.

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Recently, people (Pils̀niak, Imrich Kalinowski, Lehner) have looked at action of $Aut(\Gamma$ on edges instead, denoted $D'(\Gamma)$ or $ED(\Gamma)$.Easier generally.Started work a few months ago with Monika Pilsniak on $ED(M)$ for maps M. Immediately encountered problems multiple edges and loops arise naturally and the action of $Aut(M)$ on edges can be unfaithful.
Found out later Jozef Šìràň and Cai Heng Li have a paper from 2005 "Regular maps that do not act faithfully on vertices, edges, or faces"

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Then $Aut(M) = C_2 \times C_2$, but clearly edge unfaithful and face unfaithful since only one of each. Moreover reflection across edge fixes everything, including the two vertices!

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Equator Example Take loop in the sphere, one vertex on equator. $Aut(M) = C_2 \times C_2$ and reflection in NS longitude fixes both faces and the only vertex and only edge (note "fix" means leaves invariant) Note that M^P is projective plane with one face, so all automorphisms fix everything.

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Note that M^P may be degenerate when M is not: for tetrahedron M^P has three faces so M^{P*} has 3 vertices and 6 edges.

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For two vertices, we already have Longitude (possible with loops and multiple edges) and Equator with two vertices. **Two Vertex Antipodal** Take antipodal map and split v into two vertices joined by an edge. Half turn around midpoint is edge unfaithful but not vertex unfaithful.

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Relations and more

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2) VU does not imply EU or FU (Multiple edge equator)

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And the algebra! We've only been thinking of pictures