On the one-dimensional family of Riemann surfaces of genus q with $4q$ automorphisms

Sebastián Reyes-Carocca

Departamento de Matemática y Estadística
Universidad de La Frontera
Temuco, Chile

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Automorphism groups

Automorphism groups of Riemann surfaces have been extensively studied, going back to Wiman, Klein and Hurwitz, among others.

Let S denote a compact Riemann surface of genus g. Classically known:

- $|\mathrm{Aut}(S)| \leq 84(g-1)$.
- in the abelian case $|\mathrm{Aut}(S)| \leq 4g+4$.
- in the cyclic case $|\mathrm{Aut}(S)| \leq 4g+2$.

General problem: to understand the extent to which the order of the full automorphism group determines the Riemann surface.

Examples:

1. Hurwitz curves are $(2,3,7)$-branched coverings of the projective line.
2. $\exists!$ Riemann surface genus g admitting an automorphism of order $4g$.
3. $\exists!$ Riemann surface genus g with $8(g+1)$ automorphisms.

Automorphism groups

A very special family

Theorem (Bujalance-Costa-Izquierdo, 2017). Assume

$$g \neq 3, 6, 12, 15, 30.$$

The Riemann surfaces of genus g admitting **exactly** $4g$ automorphisms form an equisymmetric one-dimensional family, denoted by $\mathcal{F}_g$.

Moreover, if S is a Riemann surface in $\mathcal{F}_g$ then

- its full automorphism group G is isomorphic to $\mathbf{D}_{2g}$, and
- the corresponding quotient S/G has genus zero.

Remark: This is the second possible largest order (next talk!).

Plan

Let $q \geq 5$ be a prime number. For each Riemann surface S in $\mathcal{F}_q$ we study:

- an algebraic description of S and of its automorphisms,
- a decomposition of the Jacobian variety JS,
- the possible fields of definitions of S and of JS, and
- the Shimura family associated to S.

Let S denote a Riemann surface in the family $\mathcal{F}_q$ and let

$$G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle \cong \mathbf{D}_{2q}$$

denote its full automorphism group.

Algebraic description

The quotient Riemann surface S/G has genus zero, and the associated $4q$-fold branched regular covering map

$$\pi_G : S \to S/G \cong \mathbb{P}^1$$

ramifies over four values; three ramification values marked with 2 and one ramification value marked with $2q$.

Assumption. The branch values are $\infty, 0, 1$ marked with 2 and $\lambda \in \mathbb{C} - \{0, 1\}$ marked with $2q$.

Let

$$\Omega := \mathbb{C} - \{0, \pm 1, \tfrac{1}{2}, 2, \gamma, \gamma^2\} \text{ where } \gamma^3 = -1$$

denote the set of *admissible* parameters.

Algebraic description

Then $\mathcal{F}_q$ can be understood by means of an everywhere maximal rank holomorphic map

$$h : \mathcal{F}_q \to \Omega$$

in such a way that the fibers of h agree with the Riemann surfaces in $\mathcal{F}_q$. We denote by S_λ the Riemann surface $h^{-1}(\lambda)$.

Theorem. Let $\lambda \in \Omega$. Then S_λ is isomorphic to the Riemann surface defined by the normalization of the hyperelliptic algebraic curve

$$y^2 = x(x^{2q} + 2\tfrac{1+\lambda}{1-\lambda}x^q + 1).$$

The full automorphism group of S_λ is generated by the transformations

$$r(x,y) = (\omega_q x, \omega_{2q} y) \quad \text{and} \quad s(x,y) = (\tfrac{1}{x}, \tfrac{y}{x^{q+1}})$$

where $\omega_t = \exp(\frac{2\pi i}{t})$.

The Jacobian variety

It is well-known that the dihedral group

$$G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle$$

has, up to equivalence, 4 complex irreducible representations of degree one; namely,

$$V_1 : \begin{cases} r \to 1 \\ s \to 1 \end{cases} \quad V_2 : \begin{cases} r \to 1 \\ s \to -1 \end{cases} \quad V_3 : \begin{cases} r \to -1 \\ s \to 1 \end{cases} \quad V_4 : \begin{cases} r \to -1 \\ s \to -1 \end{cases}$$

and $q - 1$ complex irreducible representations of degree two; namely,

$$V_{k+4} : r \mapsto \mathrm{diag}(\omega_{2q}^k, \bar{\omega}_{2q}^k), \;\; s \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$$

for $1 \leq k \leq q - 1$ and $\omega_t = \exp(\frac{2\pi i}{t})$.

The Jacobian variety

Lemma.

(1) The rational irreducible representations of G, up to equivalence, are:

(a) four of degree 1; namely $W_i := V_i$ for $1 \leq i \leq 4$ and

(b) two of degree $q-1$; namely

$$W_5 = \oplus_{\sigma \in G_5} V_5^{\sigma} \quad \text{and} \quad W_6 = \oplus_{\sigma \in G_6} V_6^{\sigma}$$

where G_5 and G_6 denote the Galois group associated to the extensions $\mathbb{Q} \leq \mathbb{Q}(\omega_{2q} + \bar{\omega}_{2q})$ and $\mathbb{Q} \leq \mathbb{Q}(\omega_q + \bar{\omega}_q)$ respectively, and $\omega_t = \exp(\frac{2\pi i}{t})$.

(2) The group algebra decomposition of JS_λ with respect to G is

$$JS_\lambda \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2$$

where B_j stands for the factor associated to the representation W_j.

The Jacobian variety

To compute the dimension of the factors B_j (which may be zero) we need to choose a generating vector representing the action of G on S_λ.

Lemma. Let σ be a generating vector of G of type $(2,2,2,2q)$. Then there exist integers e_1, e_2 with $e_1 - e_2$ even and not congruent to 0 modulo $2q$, such that

$$\sigma = \left(sr^{e_1}, sr^{e_2}, r^q, r^{e_1-e_2+q}\right)$$

up to the action of the symmetric group $\mathbf{S}_3$ over the first three entries.

Remark. The family $\mathcal{F}_q$ is equisymmetric: every generating vector of G of the desired type can be chosen to represent the action of G on S_λ.

The Jacobian variety

Problem. To analyze how such a choice changes the dimension of the factors arising in the group algebra decomposition of JS_λ.

Definition. Two generating vectors σ_1 and σ_2 are termed *essentially equal* with respect to the action of G on S_λ if

$$\dim_{\tau_1}(B_j) = \dim_{\tau_2}(B_j)$$

for all j, where τ_i is the geometric signature associated to σ_i.

Lemma. Each generating vector of G of type $(2,2,2,2q)$ is essentially equal to

$$\sigma_0 = (s, sr^{-2}, r^q, r^{q+2}) \quad \text{or to} \quad \sigma_1 = (sr, sr^{-1}, r^q, r^{q+2}).$$

The Jacobian variety

Proposition. Let $\lambda \in \Omega$, and consider the group algebra decomposition of JS_λ with respect to G

$$JS_\lambda \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2.$$

If τ_0 denotes the geometric signature associated to σ_0, then

$$\mathsf{dim}_{\tau_0}(B_j) = \begin{cases} 0 & \text{if } j = 0, 1, 2, 3, 6 \\ 1 & \text{if } j = 4 \\ \frac{q-1}{2} & \text{if } j = 5 \end{cases}$$

If τ_1 denotes the geometric signature associated to σ_1, then

$$\mathsf{dim}_{\tau_0}(B_j) = \begin{cases} 0 & \text{if } j = 0, 1, 2, 4, 6 \\ 1 & \text{if } j = 3 \\ \frac{q-1}{2} & \text{if } j = 5 \end{cases}$$

In particular, JS_λ contains an elliptic curve.

The Jacobian variety

Theorem. Let $\lambda \in \Omega$. The group algebra decomposition of JS_λ with respect to G does not depend on the choice of the generating vector.

Proof We only need to compare the decompositions associated to σ_0 and σ_1. These decompositions are

$$JS_\lambda \sim_{G,\sigma_0} B_4 \times B_5^2 \quad \text{and} \quad JS_\lambda \sim_{G,\sigma_1} B_3 \times B_5^2$$

respectively, showing that B_3 and B_4 are **isogenous**. We claim that, in addition, B_4 and B_5 are **equal**: the outer automorphism Φ of G

$$r \mapsto r, \quad s \mapsto sr$$

identifies σ_0 and σ_1 and identifies W_3 and W_4.

The Jacobian variety

Remark The independence of the group algebra decomposition on the choice of the generating vector is not new: it was

1. noticed by Rojas when she considered the Weyl group $\mathbb{Z}_2^3 \rtimes \mathbf{S}_3$ acting on a Riemann surface of genus three with signature $(2,4,6)$.
2. noticed by Izquierdo, Jiménez and Rojas when they studied a two-dimensional family of Riemann surfaces of genus $2n-1$ with action of $\mathbf{D}_{2n}$ with signature $(2,2,2,2,n)$.

The existence of outer automorphisms of the group is the key ingredient... however it has not been proved a general result on this respect!!

From now on, we assume the action of G on S_λ to be determined by the generating vector σ_0 and

$$JS_\lambda \sim_G B_4 \times B_5^2.$$

The Jacobian variety

Theorem. Let $\lambda \in \Omega$. Consider the subgroups

$$H_4 = \langle r^{-2}, sr^{-1} \rangle \quad \text{and} \quad H_5 = \langle s \rangle$$

of G, and the quotient Riemann surfaces E_λ and C_λ given by the action of H_4 and of H_5 on S_λ, respectively. Then

$$B_4 \sim JE_\lambda \quad \text{and} \quad B_5 \sim JC_\lambda.$$

In particular, JS_λ decomposes into a product of Jacobians as follows:

$$JS_\lambda \sim_G JE_\lambda \times JC_\lambda^2.$$

Remark. C_λ is an irregular $2q$-gonal Riemann surface of genus $\frac{q-1}{2}$. The elliptic curve E_λ is algebraically represented by

$$y^2 = x(x-1)(x-\lambda).$$

Fields of definition

Let k be a subfield of $\mathbb{C}$ and let X be an algebraic variety.

Definition. The field k is a field of definition of X if there exists $Y \cong X$ such that Y is the zero locus of polynomials with coefficients in k.

Interesting fields of definition are:

1. the field of the reals,
2. the algebraic closure of $\mathbb{Q}$, and
3. the field of moduli of X.

Real Riemann surfaces An algebraic variety is called *real* if it can be defined over the field of the real numbers; equivalently, if it admits an anticonformal involution.

Remark. $\mathcal{F}_q \subset \mathscr{M}_q$ admits an anticonformal involution whose fixed point set consists of points representing real Riemann surfaces.

Fields of definition

Theorem. Let $\lambda \in \Omega$. Then the following statements are equivalent:

(a) S_λ is a real Riemann surface.

(b) JS_λ is a real algebraic variety.

(c) $\lambda \in \{\bar{\lambda}, 1 - \bar{\lambda}, 1/\bar{\lambda}, \bar{\lambda}/(1 - \bar{\lambda})\}$

Remark. The real Riemann surfaces in the family $\mathcal{F}_q$ form three one-real-dimensional arcs.

To compactify the union of these arcs in the Deligne-Mumford compactification of $\mathscr{M}_g$, it is enough to add to $\mathcal{F}_q$ three points:

1. two nodal Riemann surfaces, and
2. the *Wiman surface* of type II

Fields of definition

We can recover part of these results:

The Riemann surfaces S_{λ_1} and S_{λ_2} are isomorphic if and only if $\lambda_2 = T(\lambda_1)$ for some

$$T \in \mathbb{G} = \left\langle z \mapsto \frac{1}{z}, z \mapsto \frac{1}{1-z} \right\rangle \cong \mathbf{S}_3. \tag{1}$$

Observe that for the exceptional values $-1, \frac{1}{2}, 2, \gamma$ and γ^2 where $\gamma^3 = -1$, the Riemann surface S_λ has more than $4q$ automorphisms.

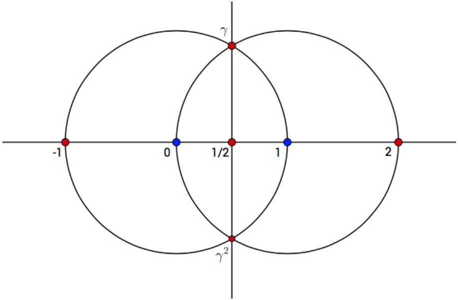
Thus, the family $\mathcal{F}_q$ is isomorphic to the quotient of the parameter space

$$\Omega = \mathbb{C} - \{0, \pm 1, \tfrac{1}{2}, 2, \gamma, \gamma^2\}$$

up to the action of $\mathbb{G}$. Namely: $\Omega \to \Omega/\mathbb{G} \cong \mathcal{F}_q \cong \mathbb{C} - \{0, 1\}$.

Fields of definition

The complex numbers $\lambda \in \Omega$ representing Riemann surfaces S_λ which are real can be represented in the diagram below; the colored red points represent Riemann surfaces with more than $4q$ automorphisms.



Fields of definition

A fundamental region for the action of $\mathbb{G}$ on Ω is given by

$$\{z \in \mathbb{C} : |z| < 1, \mathrm{Re}(z) < \tfrac{1}{2}\}$$

and, consequently, the subsets of $\mathcal{F}_q$ given by

$$\Pi(\{e^{i\theta} : \pi < \theta < \tfrac{\pi}{2}\}),\ \Pi(\{z : |z-1| = 1, |z| < 1\}) \text{ and } \Pi(]-1,0[)$$

are the three arcs in $\mathcal{F}_q$ (denoted by a_2, a_1 and b respectively)

The limit point of $\mathcal{F}_q$ which connects the arcs a_2 and b correspond to S_{-1} and therefore can be algebraically described by

$$y^2 = x(x^{2q} + 1).$$

Fields of definition

The map $(x,y) \mapsto (-\omega_{4q}x, \omega_{8q}y)$ where $\omega_t = \exp(\frac{2\pi i}{t})$, induces an isomorphism between S_{-1} and the curve

$$y^2 = x(x^{2q} - 1);$$

this is the Wiman surface of type II.

Arithmetic Riemann surfaces. An algebraic variety is called *arithmetic* if it can be defined over a number field.

Equivalence. (Belyi's theorem)

1. a Riemann surface S is arithmetic.
2. S admits a non-constant meromorphic function with three critical values.

As in the case of real Riemann surfaces, arithmetic Riemann surfaces among the Riemann surfaces in the family $\mathcal{F}_q$ can be easily identified.

Fields of definition

Theorem. Let $\lambda \in \Omega$. Then the following statements are equivalent:

(a) S_λ is an arithmetic Riemann surface.

(b) JS_λ is an arithmetic algebraic variety.

(c) λ is an algebraic complex number.

Corollary. Let $\lambda \in \Omega$ be an algebraic complex number. Then JS_λ is an arithmetic algebraic variety admitting a group algebra decomposition in which each factor is arithmetic as well.

Riemann surfaces defined over the field of moduli. The *field of moduli* $\mathcal{M}(S)$ of a compact Riemann surface S is by definition the fixed field of the group

$$\mathbb{I}(S) = \{\sigma \in \mathsf{Gal}(\mathbb{C}) : S^\sigma \cong S\}.$$

Fields of definition

Proposition. Let $\lambda \in \Omega$. Then

$$\mathbb{Q}(j(\lambda)) \leq \mathcal{M}(S) \leq \mathbb{Q}(\lambda)$$

where j denotes the Legende invariant function for elliptic curves.

- Weil: necessary conditions for S to admit its field of moduli as a field of definition.
- these conditions hold trivially if S does not have non-trivial automorphisms.
- Wolfart: if $S/\mathrm{Aut}(S)$ is an orbifold with signature of type (a, b, c) then S can be defined over its field of moduli.
- Dèbes-Emsalem: there is a field of definition of S which is an extension of finite degree of its field of moduli.

By a result of Huggins follows directly that:

Proposition. The field of moduli of S_λ is a field of definition for S_λ.

Shimura family

Let S be a compact Riemann surface of genus $g \geq 2$, and let

$$JS = (\mathscr{H}^{1,0}(S,\mathbb{C}))^* / H_1(S,\mathbb{Z})$$

be its Jacobian variety.

After fixing a symplectic basis of $H_1(S,\mathbb{Z})$ we have:

1. a period matrix $(I_g \, Z_S)$ with $Z_S \in \mathscr{H}_g$ for JS, and
2. a rational representation of $L_S := \mathrm{End}_{\mathbb{Q}}(JS) = \mathrm{End}(JS) \otimes_{\mathbb{Z}} \mathbb{Q}$

If S is hyperelliptic, then the symplectic representation

$$\rho_r : G \to \mathrm{Sp}(2g,\mathbb{Z})$$

of the automorphism group G of S induces an isomorphism

$$G \cong \mathcal{G} := \{R \in \mathrm{Sp}(2g,\mathbb{Z}) : R \cdot Z_S = Z_S\}.$$

Shimura family

We can now consider the complex submanifold of $\mathscr{H}_g$

$$\mathscr{H}_g(G) = \{Z \in \mathscr{H}_g : R \cdot Z = Z \text{ for all } R \in \mathcal{G}\}$$

consisting of those period matrices Z representing ppavs of dimension g admitting the given action of G. Clearly, $Z_S \in \mathscr{H}_g(G)$.

In the case of the action of $\mathbf{D}_{10}$ on the Riemann surfaces in family $\mathcal{F}_5$, we can be much more explicit.

Theorem. Consider the action of $\mathbf{D}_{10}$ with generating vector σ_0. There exists a three-dimensional family

$$\mathcal{A}_5(\mathbf{D}_{10}) \subset \mathcal{A}_5$$

of principally polarized abelian varieties of dimension five admitting the given group action; it is given by the period matrices in $\mathscr{H}_5$ of the following form:

Shimura family

$$\begin{pmatrix} 2(u+v+u) & -w-u & -2v & -v-w-u & -v+u \\ -w-u & -v-\frac{1}{2}w+\frac{5}{4}u & v-\frac{1}{2}u & w+\frac{1}{2}u & v-u \\ -2v & v-\frac{1}{2}u & u & v & w \\ -v-w-u & w+\frac{1}{2}u & v & u & -w \\ -v+u & v-u & w & -w & 2(u-v-w) \end{pmatrix}$$

for complex numbers u, v and w.
Furthermore, $\mathcal{A}_5(\mathbf{D}_{10})$ contains the one-dimensional family $\mathcal{F}_5$.

The automorphism group G of S can be canonically seen as a subgroup of L_S. Thus, the variety $\mathscr{H}_g(G)$ contains the complex submanifold

$$\mathbb{H}(L_S) = \text{the Shimura domain of } S$$

whose points are matrices representing ppavs containing L_S in their endomorphism algebras (the Shimura family).

Proposition. Let $\lambda \in \Omega$. The dimension of the Shimura family of each Riemann surface S_λ in $\mathcal{F}_q$ is $\frac{q+1}{2}$.

Shimura family

Given a Riemann surface S, to provide an explicit description of the elements of $\mathbb{H}(L_S)$ seems to be a difficult task.

As a simple consequence of the previous theorem, we obtain:

Corollary. Each element of the Shimura family associated to every member of the family $\mathcal{F}_5$ admits a period matrix of the form

$$\begin{pmatrix} 2(u+v+u) & -w-u & -2v & -v-w-u & -v+u \\ -w-u & -v-\frac{1}{2}w+\frac{5}{4}u & v-\frac{1}{2}u & w+\frac{1}{2}u & v-u \\ -2v & v-\frac{1}{2}u & u & v & w \\ -v-w-u & w+\frac{1}{2}u & v & u & -w \\ -v+u & v-u & w & -w & 2(u-v-w) \end{pmatrix} \tag{2}$$

for some $u, v, w \in \mathbb{C}$.

Thanks!