# Potential Versus Actual Signature Space

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Submissions still welcome for a poster session (anyone may submit, this year we are including strong undergraduate submissions). See the website for more information.

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**orbit genus** $\downarrow$ $\qquad\qquad$ $\downarrow$ $r$ **is tail length**

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A finite group $G$ acts on a compact Riemann surface $X$ of genus $g \geq 2$ with signature $[h; m_1, \ldots, m_r]$ if and only if:

I. the Riemann-Hurwitz formula is satisfied:

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right),$$

II. there exists a *generating vector* $(a_1, b_1, \ldots a_h, b_h, c_1, \ldots, c_r)$ of elements of $G$ which satisfies the following properties:

1. $G = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h, c_1, \ldots, c_r \rangle$.
2. The order of $c_j$ is $m_j$ for $1 \leq j \leq r$.
3. $\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = e_G$, the identity in $G$.

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**potential signatures** satisfy I.
**actual signatures** satisfy I. and II.

These are not always the same.

**Example**

$[0; 3, 3, 9]$ satisfies Riemann-Hurwitz for a curve of genus 2 and a group of order 9. But this signature cannot be an actual signature for abelian groups. (There's an issue with the lcm of the $m_i$. See Breuer Theorem 9.1.)

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**Example**

Take $q = p^n$ for $p$ an odd prime. Then $[0; \underbrace{2, 2, \ldots, 2}_{r>4}]$ is a potential signature for $\mathrm{SL}(2, q)$, but only one element of order 2.

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We wondered which **groups** only have a finite number of potential signatures which *fail* to be actual signatures.

We say such groups satisfy the **Finiteness Signature Condition** (FSC).

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The **order set** is:

$$\mathcal{O}(G) = \{o(g) \mid g \in G\} - \{1\}$$

$[1; n]$ will be a potential signature for every $n \in \mathcal{O}(G)$ for every group $G$.

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To create a generating vector, we need $[a, b] \in [G, G]$ and $c \in G$ with $o(c) = n$ so that $[a, b] \cdot c = e_G$. Hence $[a, b]$ must have order $n$ to create generating vector $(a, b, c)$.

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More generally, $[h; n]$ is a potential signature for every positive orbit genus $h$ and $n \in \mathcal{O}(G)$, and so there must be $a_i, b_i, c \in G$ with $o(c) = n$ so that $\prod_{i=1}^{h} [a_i, b_i] \cdot c = e_G$.

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The commutator subgroup must contain an element of every order in $\mathcal{O}(G)$.

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For each $n$ in $\mathcal{O}(G)$, the set of elements in $G$ of that order must generate the group.

Suppose $(c_1, \ldots, c_r)$ is a generating vector with $o(c_j) = n \in \mathcal{O}(G)$ and $n$ odd. Then so are:

$$(c_1, \ldots, c_r, c_r, c_r^{-1})$$

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If $(c_1, \ldots, c_r)$ is a generating vector with $o(c_j) = n$ and $n$ even and $r$ odd. Then so are:

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**Theorem (Carvacho, Paulhus, Wootton)**

*A group $G$ has the FSC if and only if the following conditions hold:*

1. *The commutator subgroup $[G, G]$ contains an element of order every $n_i \in \mathcal{O}(G)$.*
2. *For each $n_i \in \mathcal{O}(G)$, there exists an odd integer $N_i$ such that $[0; \underbrace{n_i, \ldots, n_i}_{N_i}]$ is an actual signature.*

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The proof is to show, given these conditions, the orbit genus of failures (potential signatures which are not actual signatures) is bounded and that the tail length of failures is bounded (by constructing appropriate generating vectors).

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## Example

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Then

$$(c_1, c_2, \underbrace{g_2^{-1}, g_1^{-1}, g_3}_{c_2{}^{-1}}, \underbrace{g_3^{-1}, g_2, g_2, g_1^{-1}}_{c_1{}^{-1}})$$

is a generating vector for signature

$$[0; n_1, n_2, \underbrace{n_3, \ldots, n_3}_{7}].$$

**Theorem (Carvacho, Paulhus, Wootton)**

*If a group G satisfies the FSC, then it is either a non-abelian p-group, or a perfect group (commutator subgroup is the whole group).*

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Since the commutator subgroup must contain elements of every order in $\mathcal{O}(G)$, any group satisfying FSC must be non-abelian.

What if we have a group $G$ which is not a $p$-group but satisfies FSC?

$p$ and $q$ two distinct primes in $\mathcal{O}(G)$.

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Same argument for the prime $q$ implies $G/[G,G]$ is elementary abelian of order $q^\ell$, some $\ell$. So $G/[G,G]$ must be trivial, hence $G$ is perfect. ■

Not all perfect groups satisfy FSC!

See $\mathrm{SL}(2, q)$ above when $q$ is odd.

Not all non-abelian $p$-groups satisfy FSC!

**Example**

Group $(27, 4)$ is $\langle a, b \mid a^9 = b^3 = e, bab^{-1} = a^4 \rangle$. Its commutator subgroup only has elements of order 3. The 8 elements of order 3 only generate a subgroup of order 9.

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**Example**

Group $(243, 26)$ has 170 elements of order 3 and 72 of order 9. The commutator subgroup contains elements of both non-trivial orders. But the elements of order 9 generate a subgroup of order 81.

But some $p$-groups do satisfy FSC!

**Corollary**

*For p odd, any non-abelian p-group of exponent p satisfies FSC.*

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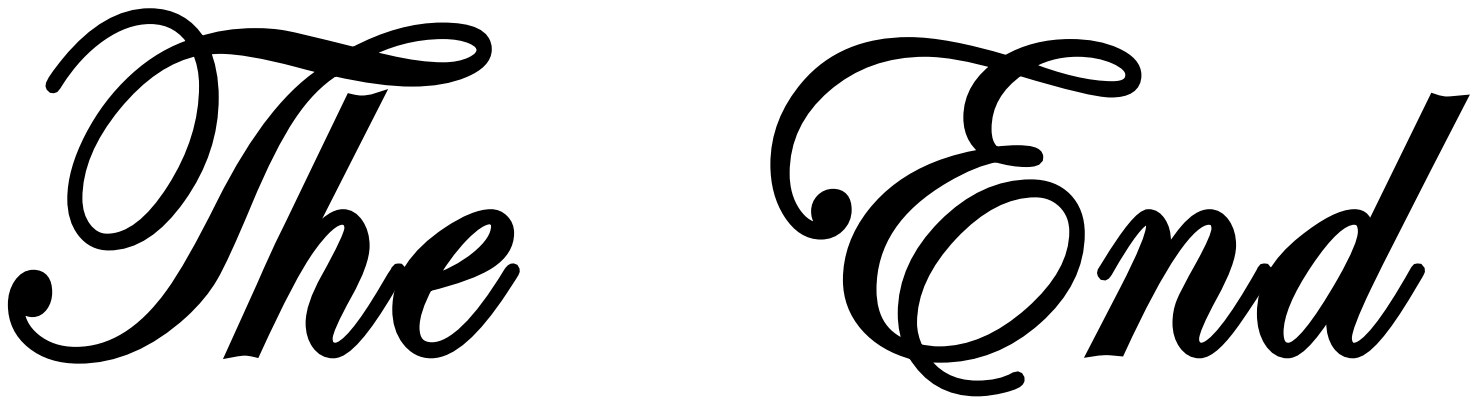
**Example**

Group $(243, 28)$ has a commutator subgroup with elements of both order 9 and 3. And the group can be generated by the elements of order 9, as well as generated by the elements of order 3.

And some perfect groups do satisfy FSC!

**Example**

PSL$(2, q)$ for all appropriate $q$ from 4 through 27.


The End