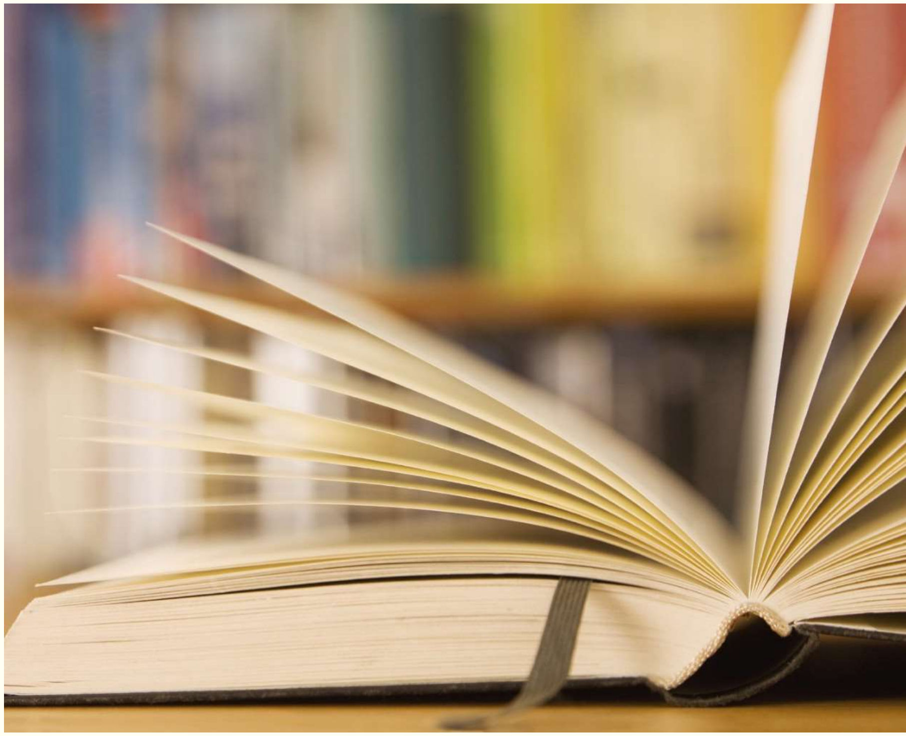
BOUNDING SKELETAL SIGNATURE SPACES



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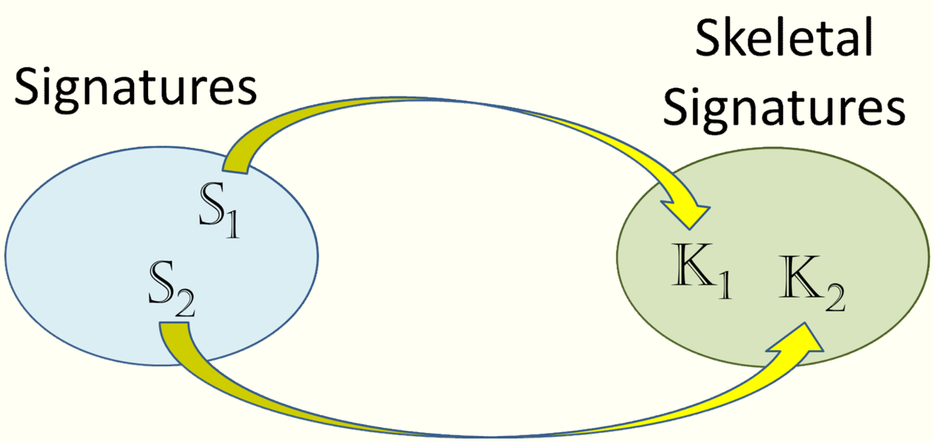
Motivation and Goals

Definition: For a given signature $(h; m_1, m_2, \ldots, m_r)$, its skeletal signature is the point (h,r)

- Can be represented as lattice points on a plane
- Need to be careful about losing signatures in the mapping
- Groups with no more than 2 elements of nontrivial order have a bijection between the signatures and the skeletal signatures
- There are three families of groups with this property



Motivation and Goals

- Discover upper and lower bounds on the number of lattice points in the skeletal signature spaces for a fixed genus, $\sigma \geq 2$.
- An upper bound could be attained by considering potential signatures.
- A lower bound could be attained by considering potential signatures that were guaranteed a group action
- After finding these bounds we wanted to study how they changed as σ increased.

p-Groups	**pq-Groups**	**p^2-Groups**
Potential Space	Potential Space	Potential Space
Absolute Space	Absolute Space	x
End Behavior	End Behavior	x

Potential Skeletal Signature Space

Definition: The potential skeletal signature space of given genus, σ **,** Order set , O, and group order, n, is the set of points (h,r) such that there exists a tuple $(h; m_1, m_2, \ldots, m_r)$ $s.t. m_i \in O$ that satisfies the Riemann Hurwitz formula for $|G| = n$.

$\sigma = 7, \qquad O = \{2,3\} \qquad n = 6$

$$2\,\sigma - 2 = 2|G|(h-1) + |G|\sum_{i=1}^{r} 1 - \frac{1}{m_i}$$

$$2(7) - 2 = 4 = 2(6)(2-1) + 0$$

So $(2; -)$ is a Potential Signature so (2,0) is a potential skeletal signature

1. $(2; -) \longrightarrow$ (2, 0)
2. $(1; 3, 3, 3) \longrightarrow$ (1, 3)
3. $(1; 2, 2, 2, 2) \longrightarrow$ (1, 4)
4. $(0; 3, 3, 3, 3, 3, 3) \longrightarrow$ (0, 6)
5. $(0; 2, 2, 2, 2, 2, 2, 2, 2) \longrightarrow$ (0, 8)
6. $(0; 2, 2, 2, 2, 3, 3, 3) \longrightarrow$ (0, 7)

Potential Skeletal Signature Spaces

- For this family $O = \{p\}$ and $|G| = p^n$
- To be a potential signature it must satisfy:

$$2\,\sigma\ -2 = 2|G|(h-1) + |G|\sum_{i=1}^{r} 1 - \frac{1}{m_i}$$

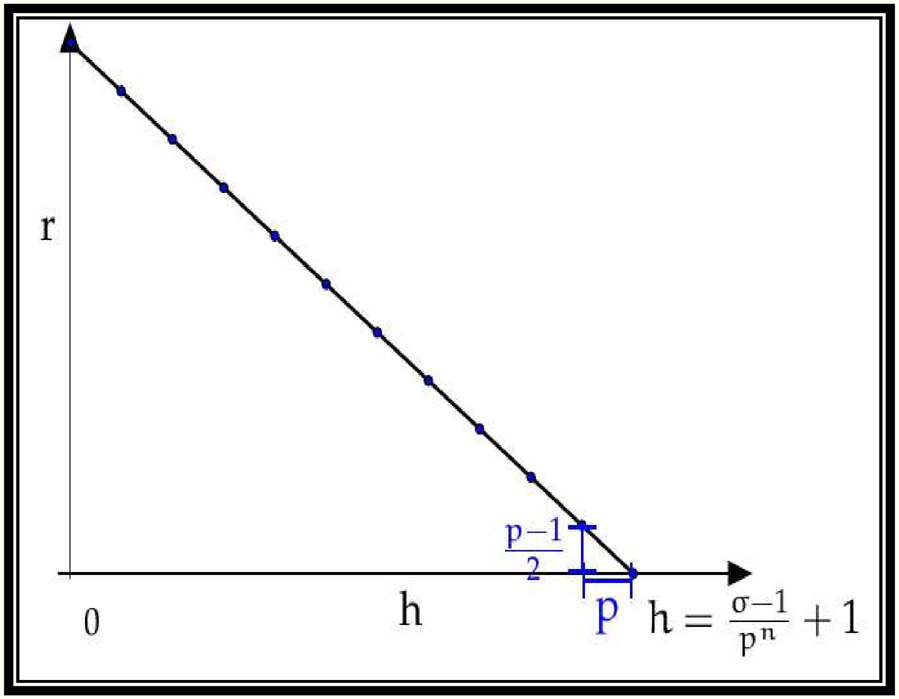
- This becomes:

$$2\,\sigma\ -2 = 2p^n(h-1) + p^n\sum_{i=1}^{r} 1 - \frac{1}{p}$$

$$\frac{\sigma-1}{p^{n-1}} = p(h-1) + r\frac{(p-1)}{2}$$

- We can see that:

$$h = h_1 + k\frac{(p-1)}{2}$$
$$r = r_1 - kp$$

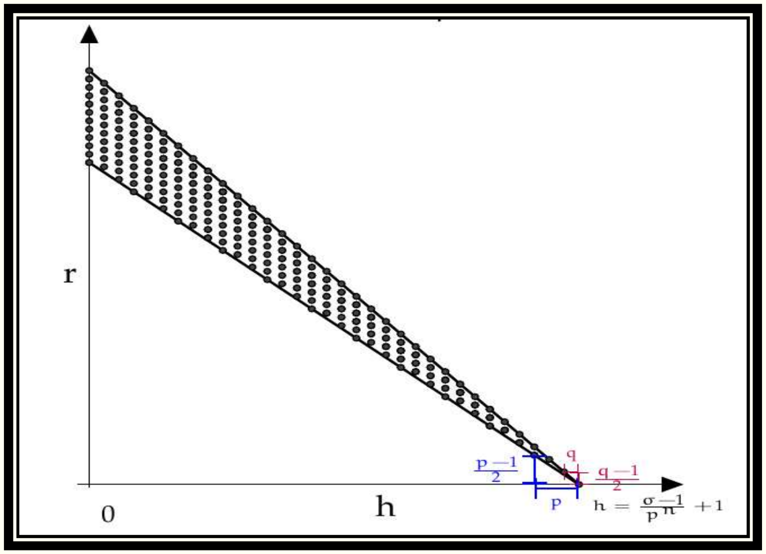


Potential Skeletal Signature Spaces

Using a similar argument for pq and p^2 groups we found

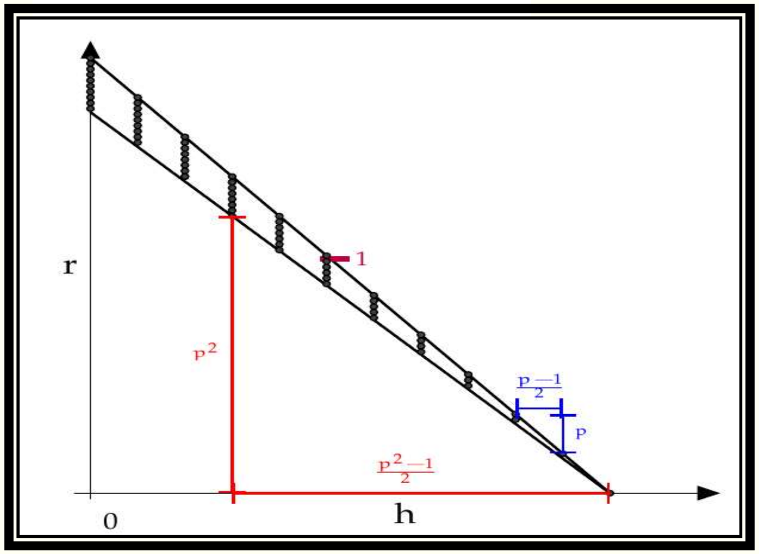
pq-Groups

$$h = h_1 + k_1\frac{p-1}{2} + k_2\frac{q-1}{2}$$
$$s = s_1 - k_1 p$$
$$t = t_1 - k_2 q$$



p^2 -Groups

$$h = h_1 + k_2\frac{p-1}{2}$$
$$s = s_1 + k_1(p+1) - k_2 p$$
$$t = t_1 - k_1 p$$

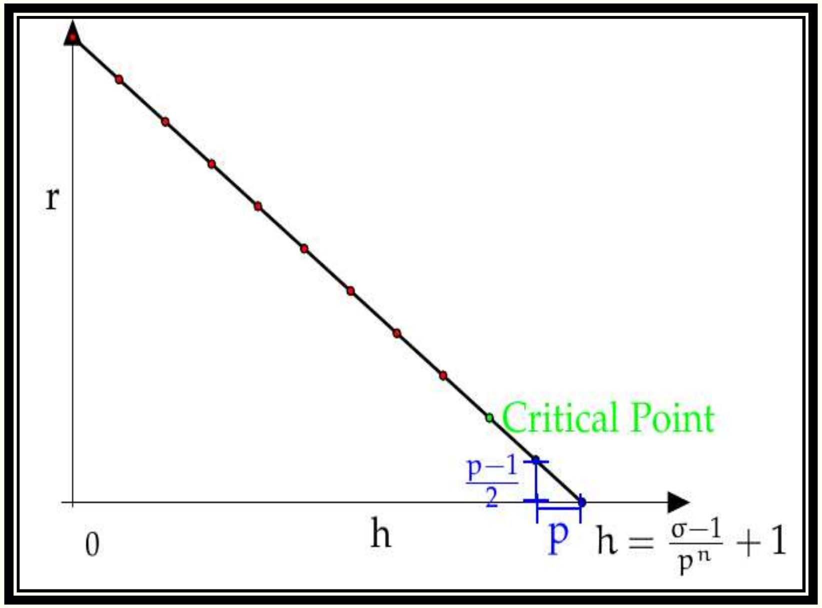


Absolute Skeletal Signature Spaces

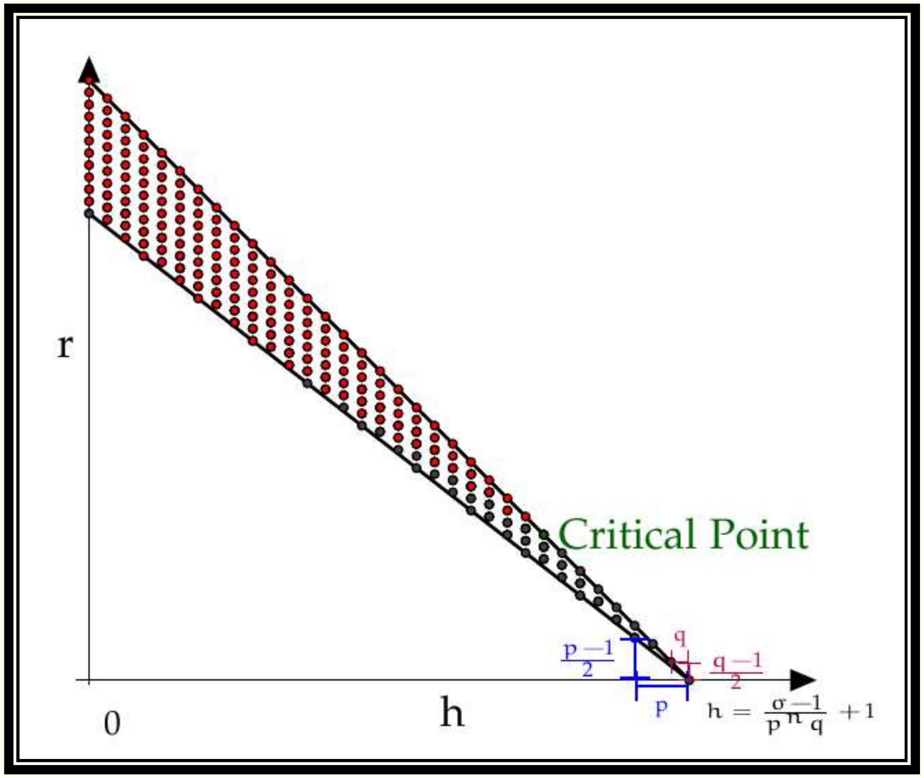
Definition: The Absolute space is the subset of the Potential Space such that a group action exists for all possible groups of that order and order set

p-Groups

- To generate a group of order p^n at least n elements are needed.
- A generating vector is guaranteed if $2h + r \geq n + 1$ and $r \neq 1$
- The exception $r = 1$ only occurs when $\frac{\sigma-1}{p^{n-1}} \equiv \frac{p-1}{2} \, mod \, p$
  - Consequence of Riemann Hurwitz Formula
- h is bounded above by $h < \frac{\sigma-1}{p^{n-1}} - \frac{n(p-1)}{2} + p$



Absolute Skeletal Signature Space



pq-Groups

- To generate a group of order $p^n q$ at least n+1 elements are needed with one being order q
- A generating vector is guaranteed if $t \geq 2 \ \vee (h > 0 \wedge t = 0)$ and $2h + s + t > n + 2$
- The exception $h = 0 \wedge t = 0$ occurs only

when $\frac{\sigma - 1}{p^{n-1} q} \equiv -1 \; mod \, \frac{p-1}{2}$

- If t=1 then $\frac{\sigma-1}{p^{n-1}} \equiv \frac{p(q-1)}{2} mod \; q$ the converse is true given

$$\frac{\sigma - 1}{p^{n-1}} \geq \frac{1}{2}(q(p-3)(p-1) + p(q-1))$$

- h is bounded above by

$$h < \frac{\sigma - 1}{p^n} + s\frac{(q-p)}{p} + 1 - n\frac{q-1}{2}$$

End Behavior ($\sigma \to \infty$)

- The potential space is equal to the absolute space if and only if

$$\sigma > \frac{p^n n(p-1) + 2p(p^{n-1}+1) - 2p^{n+1} - 2}{2(p-1)}$$

- With the one family of exceptions when $\frac{\sigma-1}{p^{n-1}} \equiv \frac{p-1}{2}\ mod\ p$ in this case (h,1) is excluded

- The potential space is equal to the absolute space if and only if

$$\sigma > \frac{n(q-1)p^n q + 2q - 2}{2(q-1)}$$

- With two families of exceptions
  - When $\frac{\sigma-1}{p^{n-1}q} \equiv -1\ mod\ \frac{p-1}{2}$ then the point where h=0 and t=0 is excluded
  - If $\frac{\sigma-1}{p^{n-1}q} \equiv \frac{p(q-1)}{2}\ mod\ q$ check that $\frac{\sigma-1}{p^{n-1}} \geq \frac{1}{2}(q(p-3)(p-1) + p(q-1))$.

Future Projects

- Establish a clear structure of the absolute space for p^2-Groups

- Generalize skeletal signatures to dimension 3+ in order to find new families whose signature space is preserved under mapping

- Use these structures to develop geometric arguments for previous results.

Thanks and Ackowledgements

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