One-dimensional families of Riemann surfaces of genus g with 4g+4 automorphisms

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joint work with A. Costa

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Given an orientable, closed surface X of genus $g \geq 2$ The

equivalence:

$(X, \mathcal{M}(X)$, complex atlas) ($\mathcal{M}(X) = \langle x, y \rangle$, $p(x, y) = 0$, the field of meromorphic functions on X)

$X \equiv \frac{\mathbb{H}}{\Delta}$, with Δ a (cocompact) Fuchsian group
Δ discrete subgroup of $PSL(2, \mathbb{R})$

$(X, \mathcal{M}(X)$, complex curve) ($\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y)$, the field of rational functions on X)
The curve X given by the polynomial $p(x, y)$ and the meromorphic function $x : X \to \widehat{\mathbb{C}}$.

Some classic results on automorphisms of curves of genus $g \geq 2$:

- There is a unique curve having an automorphism of order $4g+2$: Wiman's curve of type I $y^2 = (x^{2g+1} - 1)$, with autom. gr. $G = C_{4g+2}$,
- Except for $g = 3$, there is a unique curve having an automorphism of order $4g$: Wiman's curve of type II $y^2 = x(x^{2g} - 1)$, and autom. gr. $G = C_{4g} \rtimes_{2g-1} C_2$,
  In genus two the group is $G_2 = GL(2,3)$. The exception in genus 3 is Picards's curve $y^3 = (x^4 - 1)$, and gr. $G = C_{12}$
- The largest number,of automorphisms, of type $ag + b$, of a curve in all genera is $8g + 8$. For genera $g \equiv 0, 1, 2 \bmod 4$ there is a unique curve: Accola-Maclachlan's curve $y^2 = x(x^{2g+2} - 1)$, and gr. $G = (C_{2g+2} \times C_2) \rtimes C_2$.
  For genera $g \equiv 3 \bmod 4$ there is one more curve: Kulkarni curve $y^{2g+2} = x(x-1)^{g-1}(x+1)^{g+2}$, and gr.
  $G = \langle x, y : x^{2g+2} = y^4 = (xy)^2 = 1;\ y^2xy^2 = x^{g+2} = 1 \rangle$

Wiman 1895, Accola 1968, Maclachlan 1969, Kulkarni 1991, 1997

Kulkarni showed that, if the Riemann surfaces in a family (of RS of genus g) have at least $4g-4$ automorphisms, then the Teichmüller dimension of the family is 0 or 1.

We will see that he maximal number $ag+b$ of automorphisms of equisymmetric and uniparametric families of Riemann surfaces appearing in all genera is $4g+4$.
For all genera $g \geq 2$ there is an equisymmetric, uniparametric family $\mathcal{A}_g$ of Riemann surfaces with autom. gr. $D_{g+1} \times C_2$.
For an infinite number of genera this is the only family.
Accola-Maclachlan curve is (in the closure of) the family.

An equisymmetric family in the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is the subset of $\mathcal{M}_g$ formed by Riemann surfaces with a fixed automorphism group, acting in a given topological way.

Broughton 1990, Acosta-I 2018

Fuchsian Groups

Δ (cocompact) discrete subgroup of $PSL(2,\mathbb{R})$

A (compact) Riemann Surface (Orbifold) of genus $g \geq 2$
$X = \frac{\mathbb{H}}{\Delta}$
Δ has presentation:
generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
relations: $x_i^{m_i}, i = 1 : r,\ x_1 ... x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
x_i: generator of the maximal cyclic subgroups of Δ
$X = \frac{\mathbb{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
Δ is the orbifold-fundamental group of X.

Area of Δ: area of a fundamental region P

$\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$

X hyperbolic equivalent to $P/\langle\text{pairing}\rangle$

Poincaré's Th: $\Delta = \langle\text{pairing}\rangle$

But from now on $\mu(\Delta) = (2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$, **reduced area**.

Riemann-Hurwitz Formula: If Λ is a subgroup of finite index, N, of a Fuchsian group Δ, then $N = \frac{\mu(\Lambda)}{\mu(\Delta)}$

RUT: Any Riemann surface of genus $g \geq 2$ is uniformized by a surface Fuchsian group

$\Gamma_g = \langle a_1, b_1, ..., a_g, b_g\, ;\, a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}\rangle$

Groups of Automorphisms

G finite group of automorphisms of $X_g = \mathbb{H}/\Gamma_g$, Γ_g a surface Fuchsian group iif there exist
Δ Fuchsian group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$
θ is the monodromy of the regular covering $f : \mathbb{H}/\Gamma_g \to \mathcal{H}/\Delta$

$$
\begin{array}{ccccc}
 & & \mathbb{H} & & \\
 & \swarrow & & & \\
X/ = \mathbb{H}/\Gamma_g & & \downarrow & & \Delta\text{: lifting to } \mathbb{H} \text{ of } G \\
 & \searrow & & & \\
 & & X/G = \mathbb{H}/\Delta & &
\end{array}
$$

A morphism $f : X = \mathbb{H}/\Lambda \to Y = \mathbb{H}/\Delta$, X, Y compact Riemann orbifolds, group inclusion $i : \Lambda \to \Delta$
Singerman 71: Covering f determined by monodromy $\theta : \Delta \to \Sigma_N$ and $\Lambda = \theta^{-1}(STb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f $\leftrightarrow$ copy of fund. polygon for Δ)

Teichmüller and Moduli Spaces

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2,\mathbb{R}) \mid \sigma \mathit{injective},\ \sigma(\Delta) \text{ discrete } \}/PSL(2,\mathbb{R})$
Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h - 3 + r$, diffeomorphic to a ball of dim $6h - 6 + 2r$.

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ embedding
Γ_g surface Fuchsian group $\quad \Gamma_g \leq \Delta \quad \mathcal{T}_\Delta \subset \mathcal{T}_{\Gamma_g} = \mathcal{T}_g$

G finite group $\quad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \mid g[\sigma] = [\sigma]\, \forall g \in G\} \neq \emptyset$
$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Mapping class group $M^+(\Delta) = Out(\Delta) = \frac{Diff^+(\mathcal{H}/\Delta)}{Diff_0(\mathcal{H}/\Delta)}$
$\Delta = \pi_1(\mathcal{H}/\Delta)$ as orfibold
$M^+(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta$
$\mathcal{M}_\Delta = \mathcal{T}_\Delta / M^+(\Delta)$

Surfaces with Non-Trivial Automorphisms

Marked surface $\sigma(X) \in \mathcal{T}_g$ and $\beta \in M_g^+$,
$\beta[\sigma] = [\sigma] \quad \Leftrightarrow \quad \gamma \in PSL(2\mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$
γ induces an automorphism of the RS $[\sigma(X)]$, $Stb_{\mathcal{M}_g}[\sigma] = Aut([\sigma(X)])$
Action: $\theta : \Delta \to Aut(X_g) = G$, $ker(\theta) = \Gamma_g$
Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv. actions of G iff θ_1, θ_2 equiv under $Out(\Delta) \times Aut(G)$,
(G, θ), determines the **symmetry** of X
X_g, Y_g equisymmetric if $Aut(X_g)$ conjugate to $Aut(Y_g)$
($Aut(X_g)$: **full automorphism group**)

Broughton (1990): **Equisymmetric Stratification**
$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ is } G\}$
$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ contains } G\}$
$\mathcal{M}_g^{G,\theta}$ smooth, connected, locally closed alg. var. of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^{G,\theta}$

Signatures for Maximal Order of Automorphism Groups

Let a, b be two integers with $a > 0$. We say (a, b) is **admissible** if for all $g \geq 2$ there is an equisymmetric, uniparametric family of Riemann surfaces of genus g and with automorphism group of order $ag + b$. Let $\mathbb{A}$ be the family of admissible (a, b). Consider the lexicographic order on $\mathbb{A}$

Claim: $\max \mathbb{A} = (\mathbf{4}, \mathbf{4})$

We know that $\max \mathcal{A} \geq (4, 0)$, Bujalance-Costa-I 2017

The signatures $s_{(a,b)}$ such that (a, b) may be maximal are:

(0; 2,2,2,3),(0;2,2,2,4),(0;2,2,2,6),(0;2,2,3,3) and (0;2,2,2,g+1)

Riemann Hurwitz formula tells us:

$$(ag + b)\mu(s_{(a,b)}) = |G|\,\mu(s_{(a,b)}) = 2(g - 1)$$

Since $\dim \mathcal{T}(s_{(a,b)}) = 1$, then $s_{(a,b)} = (0; m_1, m_2, m_3, m_4)$ or $(1; m)$. If (a, b) is maximal $(a, b) \geq (4, 0)$, then $s_{(a,b)}$ must be of the form either $(0; 2, 2, 2, n])$, $n \geq 3$ or $(0; 2, 2, 3, n])$ with $3 \leq n \leq 5$.

1. For $s_{(a,b)} = (0; 2, 2, 2, 3)$ one gets $\mu(s_{(a,b)}) = \frac{1}{6}$ and $(a, b) = (12, -12)$.
2. For $s_{(a,b)} = (0; 2, 2, 2, 4)$; $\mu(s_{(a,b)}) = \frac{1}{4}$ and $(a, b) = (8, -8)$.
3. For $s_{(a,b)} = (0; 2, 2, 2, 6)$ and $(0; 2, 2, 3, 3)$; $\mu(s_{(a,b)}) = \frac{1}{3}$ and $(a, b) = (6, -6)$.

4.For $s_{(a,b)} = (0; 2, 2, 2, m)$ with $m = 5, 7, \ldots, 10$ we have respectively $\mu(s_{(a,b)}) =, \frac{3}{10} \frac{5}{14}, \frac{3}{8}, \frac{7}{18}, \frac{2}{5}$, but they are non-admissible pairs.
Again for $s_{(a,b)} = (0; 2, 2, 3, n)$, $n = 4, 5$, one has non-admissible pairs.
6. If $s_{(a,b)} = (0; 2, 2, 2, m)$ with $m > 10$, then $\mu(s_{(a,b)}) \geq \frac{9}{22}$ and:

$$2g - 2 = (ag + b)\mu(s_{(a,b)}) \geq (ag + b)\frac{9}{22}$$

Hence $\mu(s_{(a,b)}) \leq \frac{44}{9}(g - 1)$, so $(a, b) < (5, b')$ for any integer b'.

7. If we have a pair $(4, b)$, $b > 0$, with $s_{(4,b)} = (0; 2, 2, 2, m)$, there is a cyclic group in G of order m with m divides $4g + b$: $4g + b = km$:

$$2g - 2 = (4g + b)\mu(s_{(a,b)}) = (4g + b)(\frac{1}{2} - \frac{1}{m}) = 2g + \frac{b}{2} - k$$

hence $k = \frac{b}{2} + 2$ and b is an even integer.
We have $4g + b = (\frac{b}{2} + 2)m$. For the case $g = 2$ we have $\frac{b}{2} + 2$ divides $b + 8$ then $\frac{b}{2} + 2$ divides 4 and $b = 4$. Thus a pair $(4, b), b > 0$, is admissible only if $b = 4$ with signature $s_{(4,4)} = (0; 2, 2, 2, g + 1)$.

The pairs $(12,-12)$, $(8,-8)$, $(6,-6)$ are not admissible.
Example: case $(12,-12)$, with signature $(0;2,2,2,3)$.
Assume that the pair $(12,-12)$ is admissible, then for all $g \geq 2$ there is a family $\mathcal{F}_g$ of RS where each $X = \mathbb{H}/\Gamma_g \in \mathcal{F}_g$ has a group $G_g \leq Aut(X)$ with $|G_g| = 12(g-1)$ such that $X/G_g = \mathbb{H}/\Delta$ with $s(\Delta) = (0;2,2,2,3)$.
Consider $g \equiv 0 \bmod 3$ with $g-1$ a prime, $g \geq 12$.
There is a unique cyclic group C_{g-1} in G_g; the group C_{g-1} acts freely of X. Then there is a surface Fuchsian group Λ such that:

$$X = \mathbb{H}/\Gamma \to X/C_{g-1} = \mathbb{H}/\Lambda \to X/G_g = \mathbb{H}/\Delta$$

The surface X/C_{g-1} has genus 2 and the covering $\mathbb{H}/\Lambda \to \mathbb{H}/\Delta$ is a normal covering with group of automorphisms $D_6 = G_2$. The monodromy $\theta : \Delta \to D_6 = \left\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \right\rangle$ is (up to automorphisms):

$$\theta(x_1) = (s_1 s_2)^3, \theta(x_2) = s_1, \theta(x_3) = s_2, \theta(x_1) = (s_1 s_2)^2$$

The covering $X \to X/G_g$ is a regular covering, with group of automorphisms G_g an extension of C_{g-1} by D_6. The possible extensions are:

$$G_g = C_{g-1} \times D_6, G_g = C_{g-1} \rtimes_2 D_6 = D_{6(g-1)}, G_g = C_{g-1} \rtimes_1 D_6.$$

If $C_{g-1} = \langle c : c^{g-1} = 1 \rangle$ and $D_6 = \langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \rangle$, the group $C_{g-1} \rtimes_1 D_6$ satisfies $s_1 c s_1 = c^{-1}$ and $s_2 c s_2 = c$.
The covering $X = \mathbb{H}/\Gamma_g \to X/G_g$ has a monodromy $\omega : \Delta \to G_g$. But ω cannot exist for any of the three extensions G_g.
Case 1. $G_g = C_{g-1} \times D_6$: This group is not generated by elements of order two, but, $\omega(\Delta)$ is generated by $\omega(x_1)$, $\omega(x_2), \omega(x_3)$, of order two.
For $G_g = C_{g-1} \rtimes_i D_6$, $i = 1, 2$ we have that ω is as follows:
$\omega(x_1) = c^{j_1}(s_1 s_2)^3, \omega(x_2) = c^{j_2} s_1, \omega(x_3) = c^{j_3} s_2$.
Case 2. $G_g = D_{6(g-1)}$. Since $\omega(x_1)$ has order two $j_1 = 1$. Now $\omega(x_2)\omega(x_3)$ must generate C_{g-1} but then $\omega(x_4) = (\omega(x_1)\omega(x_2)\omega(x_3))^{-1}$ have order $3(g-1) \neq 3$.
Case 3. $G_g = C_{g-1} \rtimes_1 D_6$. In this case $s_1 c s_1 = c^{-1}$ and $s_2 c s_2 = c$, then $j_3 = 1$. And again if $\omega(x_2)\omega(x_3)$ generate C_{g-1} then $\omega(x_4)$ does not have order 3.

Family with Automorphism Group $D_{g+1}2 \times C_2$

For every $g \geq 2$ there is an equisymmetric and uniparametric family $\mathcal{A}_g$ of Riemann surfaces of genus g such that if $X \in \mathcal{A}_g$, $D_{g+1} \times C_2 \leq \mathrm{Aut}(X)$, the regular covering $X \to X/D_{g+1} \times C_2$ has four branched points of order $2, 2, 2$ and $g+1$ and $X/(D_{g+1} \times C_2)$ is the Riemann sphere.

This family is given by the monodromy

$$\omega : \Delta \to D_{g+1} \times C_2 = \left\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^{g+1} = 1 \right\rangle \times \left\langle t : t^2 = 1 \right\rangle$$

defined by:

$$\omega(x_1) = t, \omega(x_2) = ts_1, \omega(x_3) = s_2, \omega(x_4) = s_2 s_1$$

The surfaces in $\mathcal{A}_g$ are hyperelliptic. The hyperelliptic involution corresponds the generator t of $D_{g+1} \times C_2$.
Accola-Maclachlan surface is in the (closure of) family $\mathcal{A}_g$.

A Second Family

Assume $g \equiv -1 \bmod 4$. There is an equisymmetric uniparametric family $\mathcal{K}_g$ of Riemann surfaces such that there is G of order $4g+4$, such that $G \leq \mathrm{Aut}(X)$ for all $X \in \mathcal{K}_g$ and G is isomorphic to:

$$H_g = \left\langle \begin{array}{c} t, b, s : t^{g+1} = b^4 = s^2 = 1;\ (bs)^2 = (bt)^2 = 1; \\ b^2 = t^{\frac{g+1}{2}}; sts = t^{\frac{g-1}{2}} \end{array} \right\rangle$$

and $X/G = \mathbb{H}/\Delta$ with signature of Δ: $(0; 2, 2, 2, g+1)$.
Their automorphism groups are isomorphic to the central product of D_4 with C_{g+1}. The common center of the group is $C_2 = \langle b^2 \rangle$, with no fixed points.
The surfaces in $\mathcal{K}_g$ are non-hyperelliptic.
Kulkarni surface (with $8g+8$ automorphisms) is in the (closure of) family $\mathcal{K}_g$.

The real Riemann surface $\mathcal{A}_g$ has an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b, corresponding the real Riemann surfaces in the family. The topological closure of $\mathcal{A}_g$ in $\widehat{\mathcal{M}_g}$ has an anticonformal involution with fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) which is a closed Jordan curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of three points, two nodal surfaces and the Accola-Maclachlan surface of genus g.

The real Riemann surface $\mathcal{K}_g$ has an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b, corresponding the real Riemann surfaces in the family. The topological closure $\widehat{\mathcal{K}_g}$ of $\mathcal{K}_g$ in $\widehat{\mathcal{M}_g}$ has an anticonformal involution with fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) which is a closed curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of two points, one nodal surface and the Kulkarni surface of genus g (note that the nodal surface corresponds to a node of $\widehat{\mathcal{K}_g}$ in $\widehat{\mathcal{M}_g}$.

THANK YOU