Comparing the deck group and Veech group of an origami

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Let $E$ be an elliptic curve defined over $\mathbb{Q}$ (genus 1 curve with a rational point $\infty$).

**Definition**

An origami is a pair $(C, f)$ where $C$ is a curve and $f : C \to E$ is a map branched above at most one point.
Let $S_{g,n}$ denote a genus $g$ surface with $n$ punctures. Let $\text{Mod}(S_{g,n})$ denote the mapping class group of $S_{g,n}$. Specializing to the origami case, $\pi_1(S_{1,1}) \cong F_2 = \langle a, b, c | [a, b]c = 1 \rangle$ and $\text{Mod}(S_{1,1}) \cong \text{SL}_2(\mathbb{Z})$.

If $\Delta$ is a finite index subgroup of $F_2$, denote the $F_2$-conjugacy class of $\Delta$ by $[\Delta]$.

The mapping class group $\text{SL}_2(\mathbb{Z})$ acts on the conjugacy classes of finite-index subgroups of $F_2$ and the stabilizer of $[\Delta]$ in $\text{SL}_2(\mathbb{Z})$ via the outer action is called the **Veech group**.
The Veech group of an origami is always a subgroup of $SL_2(\mathbb{Z})$ of finite index.

When $\Gamma \leq SL_2(\mathbb{Z})$, a finite index subgroup, the corresponding quotient $\mathbb{H}/\Gamma$ of the upper half plane has a natural description as a Teichmüller curve parameterizing origami.

Origami already occur implicitly in the work of Thurston and Veech. Schmithüsen gave an algorithm for finding the Veech group of an origami.
Definition

A deck transformation or automorphism of a cover \( f : C \rightarrow E \) is a homeomorphism \( g : C \rightarrow C \) such that \( f \circ g = f \).

The set of all deck transformations of \( f \) forms a group under composition, the **deck transformation group**, \( \text{Deck}(f) \).
Each deck transformation permutes the elements of each non-critical fiber. If the action is transitive on some fiber, then it is transitive on all fibers and we call the cover **regular** (or **normal** or **Galois**).

For the origami case, in his thesis, William Chen calls covers of elliptic curves **$G$-structures**, where $G$ is the deck transformation group.
If $n \geq 1$ is an integer, then there is a map

$$\pi_n : \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/n\mathbb{Z})$$

induced by the reduction modulo $n$ map $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$.

**Definition**

The **principal congruence subgroup of level** $n$ in $\Gamma = \text{SL}_2(\mathbb{Z})$ is the kernel of $\pi_n$, denoted $\Gamma(n)$. 
Explicitly, it is described by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{n}, \quad b, c \equiv 0 \pmod{n} \right\}.$$
Definition

If $H$ is a subgroup in $\Gamma = \text{SL}_2(\mathbb{Z})$, then it is called a **congruence subgroup** if there exists $n \geq 1$ such that it contains the principal congruence subgroup $\Gamma(n)$.

Definition

The **level** $\ell$ of a congruence subgroup $H$ is the lowest common multiple of all such $n$: in fact, $\Gamma(\ell) \subseteq H$ so that it is also equal to the smallest such $n$. 
Examples

Consider the subgroup

\[ \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{n} \right\}. \]

Consider the subgroup

\[ \Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{n}, c \equiv 0 \pmod{n} \right\}. \]
A group $G$ is said to have the **congruence subgroup property** if every finite index subgroup of $G$ is a congruence subgroup.
Theorem (Bass-Milnor-Serre)

*The group $\text{SL}_n(\mathbb{Z})$ has the congruence subgroup property for $n \geq 3$."

Examples of noncongruence subgroups of $\text{SL}_2(\mathbb{Z})$ (subgroups of finite index not containing some $\Gamma(n)$) were first given by Klein in 1879.
Let $G$ be a group.

We will define the derived subgroup or the commutator subgroup of $G$ as $G^{(1)} = [G, G]$. Next, take the derived subgroup of the derived subgroup, $G^{(2)} = [G^{(1)}, G^{(1)}]$. Continue the sequence using the formula:

$$G^{(n+1)} = [G^{(n)}, G^{(n)}].$$
Definition

A group is **solvable** if and only if its derived series terminates in finitely many steps at the identity element. The smallest $\ell$ for which $G^{(\ell)}$ is the trivial group is called the **derived length** or **solvable length** of the derived series.

For example, $S_n$ is not solvable for $n \geq 5$. 
Consider the presentation of the fundamental group, 
\( \pi_1(S_{1,1}) = F_2 = \langle a, b, c | [a, b]c = 1 \rangle \). Let \( < c > \) be the cyclic group generated by \( c \). Consider the group

\[
\text{Out}(\Pi) = \{ \sigma \in \text{Aut}(F_2) | \sigma(<c>) = <c> \} / \text{Inn}(F_2).
\]
Then, there is an exterior Galois representation

$$\phi_Q : G_Q \to \text{Out}(\Pi).$$

Voedvodskiĭ proved that the profinite analogue of $\phi_Q$ is injective.
Theorem (D.)

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, $\ell$ a prime, and $F = \pi_1(E \setminus \{\infty\})(\ell) = F_2(\ell)$.

Let

$$\theta_{E,\ell} : G_\mathbb{Q} \to \text{Out}(F/F^{(2)})$$

and

$$\rho_{E,\ell} : G_\mathbb{Q} \to \text{Out}(F/F^{(1)}) \cong \text{GL}_2(\mathbb{Z}_\ell).$$

Then $\text{im}\theta_{E,\ell} \cong \text{im}\rho_{E,\ell}$.

Outer Galois representations for derived length 2 $\ell$-groups $G$ do not carry more information than their abelian (derived length 1) counterparts, $\ell$-adic Tate module Galois representations.
Theorem

A theorem due to Chen and Deligne implies that for a G-structure, where G has **derived length 2**, the corresponding Veech subgroup is congruence.

Their proof proceeds by considering outer Galois representations group-theoretically.
Example: platypus

(Ornithorhynchus)

The following picture is from the paper “Some examples of isotropic $\text{SL}_2(\mathbb{R})$-invariants subbundles of the Hodge bundle” due to Matheus and Weitze-Schmithüsen.
We find the generators of the deck group $G$:

- $\sigma_a = (7, 12, 11, 10, 9, 8)(1, 2, 3, 4, 5, 6) \in S_{12}$ and
- $\sigma_b = (1, 9, 3, 7, 5, 11)(2, 8, 6, 10, 4, 12) \in S_{12}$.

- The $G$-structure has genus 4.
The deck group is $G = \langle a, b \rangle = \text{SmallGroup}(108, 38)$, which fits into an exact sequence

$$0 \rightarrow (\mathbb{Z}/3\mathbb{Z})^2 \rightarrow G \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \rightarrow 0$$

and has derived length 2.
Therefore, by Chen/Deligne, we already know that the $G$-structure will correspond to a congruence Veech subgroup of $SL_2(\mathbb{Z})$.

Using Sage, we can find that the corresponding arithmetic subgroup of $SL_2(\mathbb{Z})$, has index 48 and $\Gamma(6) \subseteq \Gamma_G$. Also $\Gamma(n) \not\subseteq \Gamma_G$ for $n < 6$, so the level of $\Gamma_G$ is 6.
Example: $L_{2,3}$

We can find generators of the deck group:

- $\sigma_a = (2, 3, 4) \in S_4$ and $\sigma_b = (1, 2) \in S_4$.
- The $G$-structure has genus $2$.
- The deck group is $G = \langle a, b \rangle = S_4$, which has derived length $3$. 
Example: \( L_{2,3} \)

The resulting Veech group \( \Gamma_G \) is an index 9 noncongruence subgroup of \( SL_2(\mathbb{Z}) \).

Remark: Chen gives tables for all \( G \) of order \( < 256 \) corresponding to noncongruence. Notably, there are examples of both congruence and noncongruence Veech subgroups associated to derived length 3 \( G \)-structures.
Example: $L_{2,3}$

The $L_{2,3}$ origami is the $m = 3$ case of a more general construction, $L_{2,m}$. 
Proposition (Schmithüsen, Hubert, Lelièvre)

The Veech group $\Gamma(L_{2,m})$ is a noncongruence subgroup of $SL_2(\mathbb{Z})$ for $m \geq 3$. 

$L(n, m) =$
In general, the deck group of $L_{2,m}$ is $S_{m+1}$, the symmetric group on $m + 1$ letters.

Recall that $S_n$ is not solvable for $n \geq 5$. 
Conjecture (Chen)

If the deck group $G$ of a $G$-structure is not solvable, then the Veech group $\Gamma_G$ is noncongruence.
Thank you.