Comparing the deck group and Veech group of an origami

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Let E be an elliptic curve defined over $\mathbb{Q}$ (genus 1 curve with a rational point ∞).

Definition

An origami is a pair (C, f) where C is a curve and $f : C \to E$ is a map branched above at most one point.

Let $S_{g,n}$ denote a genus g surface with n punctures. Let $\mathrm{Mod}(S_{g,n})$ denote the mapping class group of $S_{g,n}$. Specializing to the origami case, $\pi_1(S_{1,1}) \simeq F_2 = \langle a, b, c | [a, b]c = 1 \rangle$ and $\mathrm{Mod}(S_{1,1}) \simeq \mathrm{SL}_2(\mathbb{Z})$.

If Δ is a finite index subgroup of F_2, denote the F_2-conjugacy class of Δ by $[\Delta]$.

The mapping class group $\mathrm{SL}_2(\mathbb{Z})$ acts on the conjugacy classes of finite-index subgroups of F_2 and the stabilizer of $[\Delta]$ in $\mathrm{SL}_2(\mathbb{Z})$ via the outer action is called the **Veech group**.

The Veech group of an origami is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.

When $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, a finite index subgroup, the corresponding quotient $\mathbb{H}/\Gamma$ of the upper half plane has a natural description as a Teichmüller curve parameterizing origami.

Origami already occur implicitly in the work of Thurston and Veech. Schmithüsen gave an algorithm for finding the Veech group of an origami.

Definition

A deck transformation or automorphism of a cover $f : C \to E$ is a homeomorphism $g : C \to C$ such that $f \circ g = f$.

The set of all deck transformations of f forms a group under composition, the **deck transformation group**, $\mathrm{Deck}(f)$.

Each deck transformation permutes the elements of each non-critical fiber. If the action is transitive on some fiber, then it is transitive on all fibers and we call the cover **regular** (or **normal** or **Galois**).

For the origami case, in his thesis, William Chen calls covers of elliptic curves G-**structures**, where G is the deck transformation group.

If $n \geq 1$ is an integer, then there is a map

$$\pi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$$

induced by the reduction modulo n map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

Definition

The **principal congruence subgroup of level** n in $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is the kernel of π_n, denoted $\Gamma(n)$.

Explicitlly, it is described by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{n}, \quad b, c \equiv 0 \pmod{n} \right\}.$$

Definition

If H is a subgroup in $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then it is called a **congruence subgroup** if there exists $n \geq 1$ such that it contains the principal congruence subgroup $\Gamma(n)$.

Definition

The **level** ℓ of a congruence subgroup H is the lowest common multiple of all such n: in fact, $\Gamma(\ell) \subset H$ so that it is also equal to the smallest such n.

Examples

Consider the subgroup

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{n} \right\}.$$

Consider the subgroup

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{n}, c \equiv 0 \pmod{n} \right\}.$$

Congruence subgroup problem

Definition

A group G is said to have the **congruence subgroup property** if every finite index subgroup of G is a congruence subgroup.

Theorem (Bass-Milnor-Serre)

The group $\mathrm{SL}_n(\mathbb{Z})$ has the congruence subgroup property for $n \geq 3$.

Examples of noncongruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (subgroups of finite index not containing some $\Gamma(n)$) were first given by Klein in 1879.

Let G be a group.

We will define the derived subgroup or the commutator subgroup of G as $G^{(1)} = [G, G]$. Next, take the derived subgroup of the derived subgroup, $G^{(2)} = [G^{(1)}, G^{(1)}]$. Continue the sequence using the formula:

$$G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Definition

A group is **solvable** if and only if its derived series terminates in finitely many steps at the identity element. The smallest ℓ for which $G^{(\ell)}$ is the trivial group is called the **derived length** or **solvable length** of the derived series.

For example, S_n is not solvable for $n \geq 5$.

Consider the presentation of the fundamental group, $\pi_1(S_{1,1}) = F_2 = \langle a, b, c | [a, b]c = 1\rangle$. Let $< c >$ be the cyclic group generated by c. Consider the group

$$Out(\Pi) = \{\sigma \in \mathrm{Aut}(F_2) | \sigma(< c >) =< c >\} / Inn(F_2).$$

Then, there is an exterior Galois representation

$$\phi_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathrm{Out}(\Pi).$$

Voedvodskiĭ proved that the profinite analogue of $\phi_{\mathbb{Q}}$ is injective.

Theorem (D.)

Let E be an elliptic curve defined over $\mathbb{Q}$, ℓ a prime, and $F = \pi_1(E \setminus \{\infty\})(\ell) = F_2(\ell)$.
Let

$$\theta_{E,\ell} : G_{\mathbb{Q}} \to \mathrm{Out}(F/F^{(2)})$$

and

$$\rho_{E,\ell} : G_{\mathbb{Q}} \to \mathrm{Out}(F/F^{(1)}) \simeq \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Then $\mathrm{im}\theta_{E,\ell} \simeq \mathrm{im}\rho_{E,\ell}$.

Outer Galois representations for **derived length 2** ℓ-groups G do not carry more information than their abelian (**derived length 1**) counterparts, ℓ-adic Tate module Galois representations.

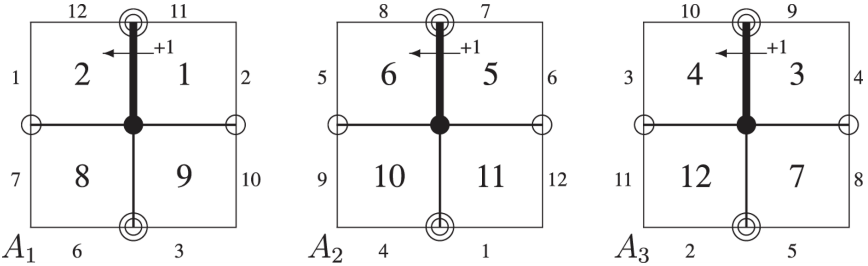
Theorem

*A theorem due to Chen and Deligne implies that for a G-structure, where G has **derived length 2**, the corresponding Veech subgroup is congruence.*

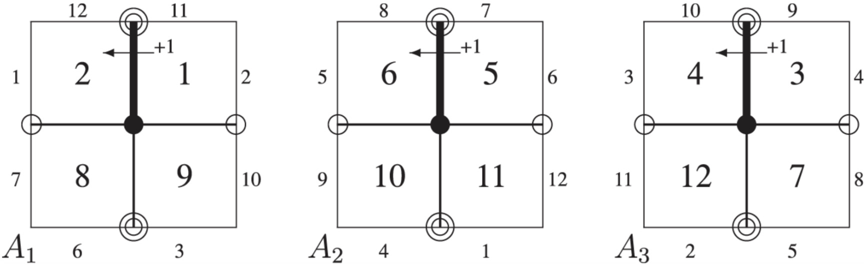
Their proof proceeds by considering outer Galois representations group-theoretically.

Example: platypus

(Ornithorhynchus)
The following picture is from the paper "Some examples of isotropic $\mathrm{SL}_2(\mathbb{R})$-invariants subbundles of the Hodge bundle" due to Matheus and Weitze-Schmithüsen.



Example: platypus



We find the generators of the deck group G:

- $\sigma_a = (7,12,11,10,9,8)(1,2,3,4,5,6) \in S_{12}$ and
- $\sigma_b = (1,9,3,7,5,11)(2,8,6,10,4,12) \in S_{12}$.
- The G-structure has genus 4.

Platypus example

The deck group is $G = \langle a, b \rangle = \mathrm{SmallGroup}(108, 38)$, which fits into an exact sequence

$$0 \to (\mathbb{Z}/3\mathbb{Z})^2 \to G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \to 0$$

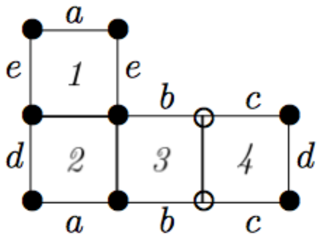
and has derived length 2.

Platypus example

Therefore, by Chen/Deligne, we already know that the G-structure will correspond to a congruence Veech subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Using Sage, we can find that the corresponding arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$, has index 48 and $\Gamma(6) \subseteq \Gamma_G$. Also $\Gamma(n) \not\subseteq \Gamma_G$ for $n < 6$, so the level of Γ_G is 6.

Example: $L_{2,3}$



We can find generators of the deck group:

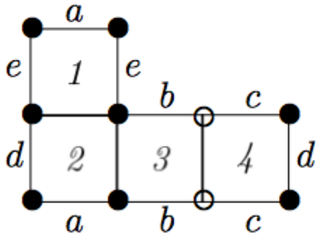
- $\sigma_a = (2,3,4) \in S_4$ and
- $\sigma_b = (1,2) \in S_4$.
- The G-structure has genus 2.
- The deck group is $G = \langle a, b \rangle = S_4$, which has derived length 3.

Example: $L_{2,3}$

The resulting Veech group Γ_G is an index 9 **noncongruence** subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Remark: Chen gives tables for all G of order < 256 corresponding to noncongruence. Notably, there are examples of both congruence and noncongruence Veech subgroups associated to derived length 3 G-structures.

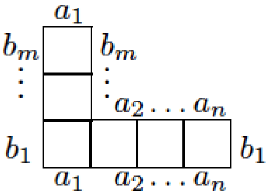
Example: $L_{2,3}$



The $L_{2,3}$ origami is the $m = 3$ case of a more general construction, $L_{2,m}$.

$L_{2,m}$

$L(n,m) =$



Proposition (Schmithüsen, Hubert, Lelièvre)

The Veech group $\Gamma(L_{2,m})$ is a noncongruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ for $m \geq 3$.

In general, the deck group of $L_{2,m}$ is S_{m+1}, the symmetric group on $m+1$ letters.

Recall that S_n is not solvable for $n \geq 5$.

Conjecture (Chen)

If the deck group G of a G-structure is not solvable, then then the Veech group Γ_G is noncongruence.

Thank you.