Dessins d'Enfants
and Topological Cyclic Actions on Surfaces

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Dessins d'Enfants

A **dessin d'enfant** is a pair (X, D), where X is an orientable, compact surface and $D \subset X$ is a finite graph such that

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Two dessins (X, D) and (X', D') are **equivalent** if there exists an orientation-preserving homeomorphism $X \to X'$ whose restriction to D induces isomorphisms as bicolored graphs.

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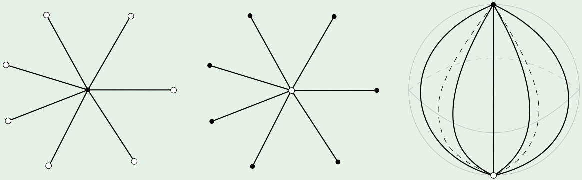
Example

Three regular dessins on the Riemann sphere with $\mathrm{Aut}(D) \cong C_7$, the cyclic group of order seven.



Triangle Groups

A **triangle group** $\Delta(\ell, m, n)$ is a group with the following presentation:

$$\Delta(\ell, m, n) = \langle x, y, z : x^\ell = y^m = z^n = xyz = 1 \rangle.$$

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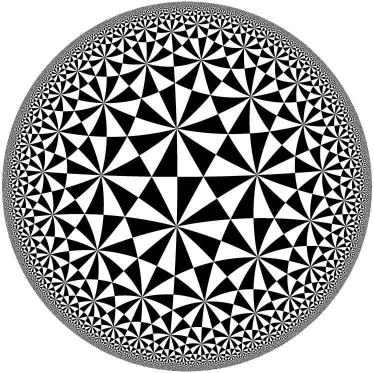

Figure: Tessellation of $\mathbb{H}$ by $\Delta(2, 3, 7)$.

Quasiplatonic Surfaces

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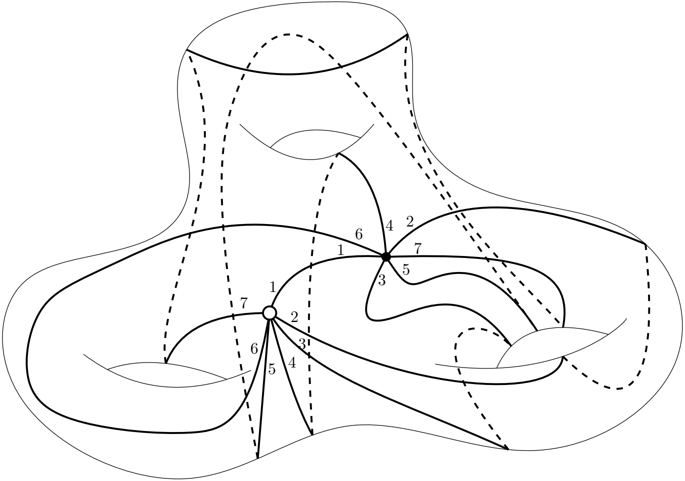
Quasiplatonic surfaces are the most symmetric of all compact Riemann surfaces.

Example of Regular Dessin on Genus Three Surface



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Moreover, $\mathrm{Aut}(D)$ acts by automorphisms on X.

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- $\eta_1 \eta_2 \eta_3 = 1$;
- $\langle \eta_1, \eta_2, \eta_3 \rangle = G$.

The tuple (η_1, η_2, η_3) is called a (n_1, n_2, n_3)-**generating vector** of G.

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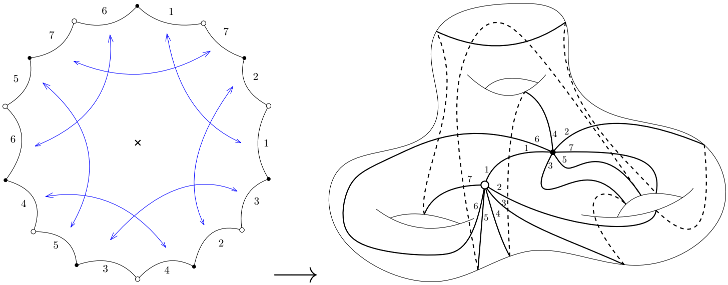
$$(\rho^2, \rho^4, \rho^1), \quad (\rho^3, \rho^3, \rho^1), \quad (\rho^1, \rho^5, \rho^1)$$

are all $(7, 7, 7)$-generating vectors for G.

Generating Vectors and Dessins

Identifying edges as shown makes a quasiplatonic surface $X \cong \mathbb{H}/\Gamma$ with $\Gamma \triangleleft \Delta(7,7,7)$ and $\Delta(7,7,7)/\Gamma \cong C_7$.



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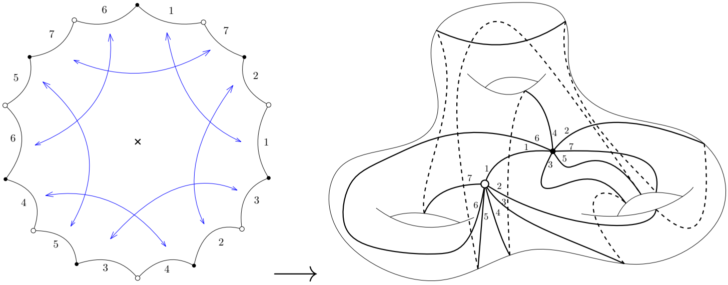

- If $\rho = (1,2,3,4,5,6,7)$ denotes rotation by $2\pi/7$ clockwise about the face-center bounded by the seven numbered edges, then the edges about the white and black vertices are described by the permutations ρ and ρ^5, respectively.

Group Acting on a Surface

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Two actions ϵ_1 and ϵ_2 are **equivalent** if $\epsilon_1(G)$ and $\epsilon_2(G)$ are conjugate in $\text{Homeo}^+(X)$.

Main Questions

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2. How does the total number of actions of C_n relate to the number of regular dessins D with $\mathrm{Aut}(D) \cong C_n$?

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2. *for n even, exactly two of n_1, n_2, n_3 must be divisible by the maximum power of two dividing n;*
3. *the Riemann-Hurwitz formula holds:*

$$g = 1 + \frac{n}{2}\left(1 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right).$$

Enumerating Topological Cyclic Actions - Benim, Wootton 2013

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Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the prime factorization of n.

Signature	T = number of distinct topological actions
(n_1, n_2, n_3)	$T = \phi(gcd(n_1, n_2, n_3)) \cdot \prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}$
(n_1, n, n)	$T = \frac{1}{2}\left(\tau_1(n, n_1) + \phi(n) \cdot \prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right)$
(n, n, n)	$T = \frac{1}{6}\left(3 + 2\tau_2(n) + \phi(n) \cdot \prod_{i=1}^{r} \frac{p_i - 2}{p_i - 1}\right)$

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- $\tau_1(n_1, n)$ = number of noncongruent, nonzero solutions to $x^2 + 2x \equiv 0 \bmod n$ where $gcd(x, n) = n/n_1$;
- $\tau_2(n)$ = number of noncongruent solutions to $x^2 + x + 1 \equiv 0 \bmod n$;
- $w \geq 0$ is an integer representing the number of primes (including multiplicity) shared in common among n_1, n_2, n_3.

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3. add up all values given by the formulas from all possible signatures for n. This number will be $QC(n)$.

Example

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Then $QC(20) = 1 + 1 + 1 + 2 + 2 = 7$.

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Then

$$QC(p) = T(p,p,p) = \begin{cases} \frac{1}{6}(p+1) & p \equiv 5 \bmod 6 \\ \frac{1}{6}(p+1) + \frac{2}{3} & p \equiv 1 \bmod 6 \end{cases} .$$

Preliminary Results

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Theorem (C, 2018)

Suppose n is even and $n \geq 8$, so that $n = 2^{a_1} \prod_{i=2}^{r} p_i^{a_i}$. Then the number of distinct topological actions of C_n on quasiplatonic surfaces is

$$QC(2^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = 2^{a_1 - 2} \left(\prod_{i=2}^{r} p_i^{a_i - 1}(p_i + 1) \right) - 1 + \begin{cases} 2^{r-2} & a_1 = 1 \\ 2^{r-1} & a_1 = 2 \\ 2^{r} & a_1 \geq 3 \end{cases} .$$

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Corollary

Let $r(C_n)$ denote the number of regular dessins with C_n as their automorphism group. Then for even $n \geq 8$,

$$QC(n) - \frac{1}{6} r(C_n) = -1 + \begin{cases} 2^{r-2} & a_1 = 1 \\ 2^{r-1} & a_1 = 2 \\ 2^{r} & a_1 \geq 3 \end{cases}.$$

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(A) First show, when $n = 2 \cdot \prod_{i=2}^{r} p_i^{a_i}$

$$QC(n \cdot p_{r+1}^{a_{r+1}}) = \left(QC(n) + 1 - 2^{r-2}\right) p_{r+1}^{a_{r+1}-1}(p_{r+1}+1) - 1 + 2^{r-1}.$$

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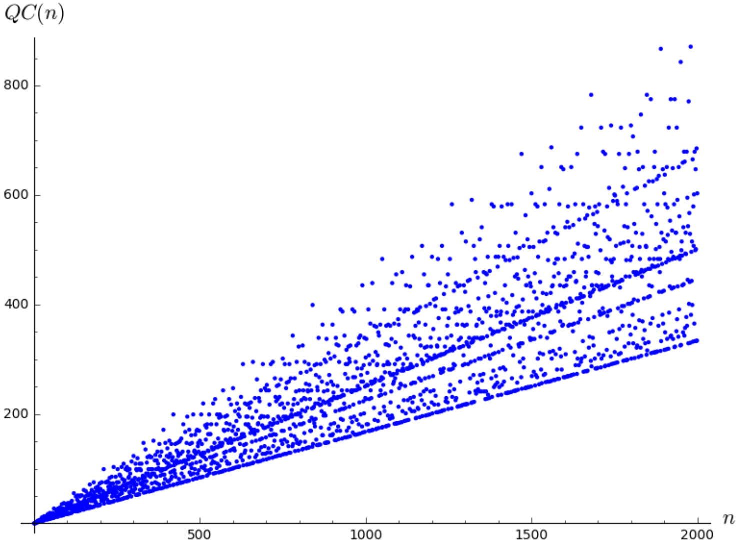
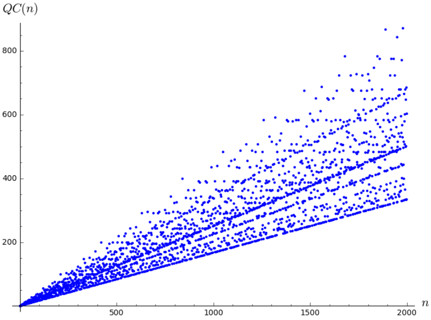
(B) Then prove, for any even $n \geq 8$,

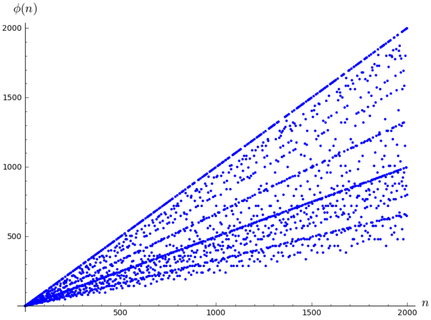
$$QC(2^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = 2 \cdot QC(2^{a_1-1} p_2^{a_2} \cdots p_r^{a_r}) + 1 + \begin{cases} 0 & 2 \leq a_1 \leq 3 \\ -2^r & 4 \leq a_1 \end{cases}.$$

$QC(n)$ Graph



$QC(n)$ versus Euler Totient Function $\phi(n)$





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- Physics applications? (string theory, Fenyman diagrams, statistical mechanics...)

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Questions? Thank you!

Group Actions and Generating Vectors

Theorem (Riemann Existence Theorem)

A group G acts topologically on X of signature (n_1, n_2, n_3) of genus $g \geq 2$ if and only if G has a (n_1, n_2, n_3)-generating vector and the Riemann-Hurwitz formula holds:

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The equivalence of G-actions on X of signature (n_1, n_2, n_3) induces an equivalence on the (n_1, n_2, n_3)-generating vectors ν of G:

Group Actions and Generating Vectors

Theorem (Riemann Existence Theorem)

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The equivalence of G-actions on X of signature (n_1, n_2, n_3) induces an equivalence on the (n_1, n_2, n_3)-generating vectors ν of G:

- Two vectors ν and ν' are **equivalent** if there exists $(w, \phi) \in \mathrm{Aut}(G) \times \mathrm{Aut}(\Delta)$ such that $\rho' = w \circ \rho \circ \phi^{-1}$, where ρ and ρ' are the corresponding surface-kernel epimorphisms of ν and ν'.

Proof Sketch - Recursive Formula A

Signatures for $n = 2\prod_{i=2}^{r} p_i^{a_i}$ are of the form

$$\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r}\right).$$

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We extend to signatures for $n \cdot p_{r+1}^{a_{r+1}}$ by multiplying each period by p_{r+1}^{t}.

Proof Sketch - Notation

Consider p^a for a prime p and a positive integer a. If $a \geq 2$, define

$$f(p^k) = \begin{cases} 1 & k = 0 \\ p^{k-1}(p-1) & 1 \leq k \leq a-1 \\ p^{a-1}(p-2) & k = a \end{cases} .$$

If $a = 1$, then define

$$f(p^k) = \begin{cases} 1 & k = 0 \\ p - 2 & k = 1 \end{cases} .$$

Proof Sketch - Two Cases

Fix parameters $k_1, \ldots, k_r$.
For a signature $\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r}\right)$ of n,

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- $\underline{\text{if } \ell_1, \ldots, \ell_r \text{ and } h_1, \ldots, h_r \text{ are not all equal}}$, then

$$T\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r}\right) = \prod_{i=2}^{r} f(p_i^{k_i}).$$

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- if $\ell_1, \ldots, \ell_r$ and $h_1, \ldots, h_r$ are all equal, then

$$T\left(p_2^{k_2} \cdots p_r^{k_r}, 2p_2^{a_2} \cdots p_r^{a_r}, 2p_2^{a_2} \cdots p_r^{a_r}\right) = \begin{cases} \frac{1}{2} \prod_{i=2}^{r} f(p_i^{k_i}) & 1 \le k_i \le a_i - 1 \text{ for some } i \\ \frac{1}{2} \prod_{i=2}^{r} f(p_i^{k_i}) + \frac{1}{2} & \text{otherwise} \end{cases}.$$

Proof Sketch - Extending The First Case

Assume $\ell_1, \ldots, \ell_r$ and $h_1, \ldots, h_r$ are not all equal.
Signatures for $n \cdot p_{r+1}^{a_{r+1}}$ consist of the following three forms:

$$\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r} p_{r+1}^{t}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r} p_{r+1}^{a_{r+1}}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r} p_{r+1}^{a_{r+1}}\right), 0 \leq t \leq a_{r+1},$$
$$\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r} p_{r+1}^{a_{r+1}}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r} p_{r+1}^{t}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r} p_{r+1}^{a_{r+1}}\right), 0 \leq t \leq a_{r+1} - 1,$$
$$\left(2^{k_1} p_2^{k_2} \cdots p_r^{k_r} p_{r+1}^{a_{r+1}}, 2^{\ell_1} p_2^{\ell_2} \cdots p_r^{\ell_r} p_{r+1}^{a_{r+1}}, 2^{h_1} p_2^{h_2} \cdots p_r^{h_r} p_{r+1}^{t}\right), 0 \leq t \leq a_{r+1} - 1.$$

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Then the total number of topological actions for this case is

$$\begin{aligned} &\prod_{i=2}^{r} f(p_i^{k_i}) \cdot \left(\sum_{t=0}^{a_{r+1}} (f(p_{r+1}^{t})) + 2 \sum_{t=0}^{a_{r+1}-1} (f(p_{r+1}^{t})) \right) \\ &= \prod_{i=2}^{r} f(p_i^{k_i}) \cdot \left(p_{r+1}^{a_{r+1}-1} (p_{r+1} + 1) \right). \end{aligned}$$

Proof Sketch - Extending the Second Case

We consider signatures for $n \cdot p_{r+1}^{a_{r+1}}$ of the following two possible extensions:

$$\left(p_2^{k_2} \cdots p_r^{k_r} p_{r+1}^{t}, 2^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}}, 2^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}}\right), 0 \leq t \leq a_{r+1}$$

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We will incur factors of $1/2$ depending on whether

- $k_i = a_i$ or $k_i = 0$ for each i
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Consider those subcases, and follow the same procedure as before...! $\square$