Topological and $\mathcal{H}^q$ Equivalence of Prime Cyclic p-gonal Actions on Riemann Surfaces

Preliminary Report

S. Allen Broughton - Rose-Hulman Institute of Technology

14 Apr 2018

Sections

Overview of sections

- Overview, introduction, and historical motivation
- $\mathcal{H}^q(S)$ and representations of $\mathrm{Aut}(S)$
- Actions and signatures - especially prime cyclic n-gonal actions
- Topological vs $\mathcal{H}^q$ equivalence
- Results and examples

Objects of study

- S, a Riemann surface of genus σ
- $\mathrm{Aut}(S)$ the automorphisms of S and $G \subseteq \mathrm{Aut}(S)$ a subgroup of interest
- $\mathcal{H}^q$ the space of globally defined q-differentials on S

History - 1

- Definition and study of Riemann surfaces and their automorphism begun last half of 19th century.
- Surfaces and automorphism linked to Fuchsian groups about that time.
- Some enumeration of automorphism groups begun around the turn of the (last century).
- Continuation of the study of the above, along with moduli space and Teichmüller space during the last and this century.
- Computer computation has dramatically helped exploration in the last 30 years.

History - 2

- Two classification schemes of automorphism groups:
  - Linear: action of automorphism group on Abelian differentials and the characters of these representations.
  - Topological: covering spaces, fundamental groups, uniformization by Fuchsian groups. Linked to moduli space and mapping class group.
- Topological classification in near perfect correspondence to Moduli space structures. There is a simple conceptual but computationally expensive method of enumeration.
- Linear classification has low genus failures: next slide.

Problem Fix?

- There are two topologically distinct actions of $\mathbb{Z}_5$ in genus 4 that are linearly equivalent.
- The actions are *conflated*.

Question

Can the conflation problem be fixed by looking at higher order differentials?

Conjecture

The answer to the question is yes.

q-differentials - 1

$q = 1$, Abelian differentials - significant in analysis and geometry of surfaces

- globally defined holomorphic differential 1-forms on a surface
- $\omega = f_1(z)dz$ in local co-ordinates
- define interesting functions via

$$F(z) = \int_{z_0}^{z} \omega$$

- used to measure moduli properties of surfaces - integrate along a homology basis of loops on the surface
- used to construct the Jacobian variety of a surface

q-differentials - 2

$q = 2$, Quadratic differentials - also have analytic and geometric applications

- $f_2(z)(dz)^2$ in local co-ordinates
- $f_2(z)(dz)^2$ is holomorphically varying homogeneous form of degree 2 on the tangent bundle $T(S)$
- $\mathcal{H}^2(S)$ may be identified with the tangent space of Teichmüller space at the point S.

q-differentials - 3

General q

- $T^*(S) =$ cotangent (line) bundle of S, i.e ., "linear forms" on vector fields
- $\mathcal{H}^q(S)$ = global sections of the q-fold tensorial line bundle $T^*(S) \otimes \cdots \otimes T^*(S)$
- $f_q(z)(dz)^q$ in local co-ordinates
- $\mathcal{H}^q(S)$ yields an interesting projective embedding of a surface, $q = 1$ is canonical embedding
- dimension over $\mathbb{C}$

$$\dim \mathcal{H}^1(S) = \sigma$$
$$\dim \mathcal{H}^q(S) = (2q-1)(\sigma - 1), q > 1$$

Representations and characters - 1

- $\mathrm{Aut}(S)$ acts on $\mathcal{H}^q(S)$ via

$$g^*(f_q(z)(dz)^q = (f_q(g^{-1}z)(dg^{-1}z)^q$$

- the representation of $G \subseteq \mathrm{Aut}(S)$ on $\mathcal{H}^q(S)$ is completely determined by its character

$$\mathrm{ch}_{\mathcal{H}^q(S)}(g) = \mathit{Trace}(g^* \text{ on } \mathcal{H}^q(S))$$

- set notation:
  - irreducible characters of G: $\chi_0, \ldots, \chi_l$
  - multiplicity inner product: $\mu_q^j = \langle \mathrm{ch}_{\mathcal{H}^q(S)}, \chi_j \rangle$

Representations and characters - 2

- write

$$\mathrm{ch}_{\mathcal{H}^q(S)} = \mu_q^0\chi_0 + \cdots + \mu_q^l\chi_l,$$

- define the column vector R_q

$$R_q^t = \begin{bmatrix} \mu_q^0 & \cdots & \mu_q^l \end{bmatrix}$$

- gather various R_q into a matrix sufficient to describe the characters for all q-differentials

$$M = \begin{bmatrix} R_1 & \cdots & R_{|G|} \end{bmatrix} = \left[\mu_q^j\right],\ j = 0, \ldots, l, q = 1, \ldots, |G|$$

Fixed points and rotation numbers

- Fixed points:
$$S^g = \{P : gP = P\}$$
- Rotation number $\epsilon(P, g)$ at fixed point in S^g:
$$dg^{-1} : T_P(S) \to T_P(S)$$
is multiplication by $\epsilon(P, g)$, an $o(g)$th root of unity.

Eichler trace formula

- Define $\lambda_q : G \to \mathbb{C}$, $q \geq 1$

$$\lambda_q(1) = (\sigma - 1)(2q - 1)$$
$$\lambda_q(g) = \sum_{P \in S^g} \frac{(\varepsilon(P,g))^q}{1 - \varepsilon(P,g)}$$

- Eichler's trace formula:

$$\mathrm{ch}_{\mathcal{H}^1(S)}(g) = 1 + \lambda_1(g) = \chi_0(g) + \lambda_1(g)$$
$$\mathrm{ch}_{\mathcal{H}^q(S)}(g) = \lambda_q(g), q > 1$$

Linear + Periodic character

- the Eichler trace formula shows a linear (depending only on G) plus periodic (depending on the action) pattern
- in formulas below $|G|$ can be replaced by $exp(G)$

Proposition

For $q > |G|$, write $q = a|G| + b$ with $1 \leq b \leq |G|$. Then,

$$\mathrm{ch}_{\mathcal{H}^q(S)} = 2a(\sigma - 1)\chi_{reg} + \mathrm{ch}_{\mathcal{H}^b(S)} - \chi_0, \quad b = 1,$$
$$\mathrm{ch}_{\mathcal{H}^q(S)} = 2a(\sigma - 1)\chi_{reg} + \mathrm{ch}_{\mathcal{H}^b(S)}, \quad b \neq 1.$$

- the periodic component nature is encoded by the multiplicity matrix, introduced earlier,

$$M = \left[\mu_q^j\right]$$

Group actions - 1

- conformal group action: a monomorphism

$$\epsilon : G \to \mathrm{Aut}(S)$$

- assume S/G has genus 0, so that the action is n-gonal
- $\pi : S \to S/G$ is branched over t points $Q_1, \ldots, Q_t$
- define $T = S/G \simeq \mathbb{P}^1$, $T^\circ = T - \{Q_1, \ldots, Q_t\}$, and $S^\circ = \pi^{-1}(T^\circ)$.
- $\pi : S^\circ \to T^\circ$ is a regular covering space with deck transformation group G.

Group actions - 2

- let $Q_0 \in T^\circ$, then

$$\pi_1(T^\circ, Q_0) = \langle \gamma_1, \ldots, \gamma_t : \gamma_1 \cdots \gamma_t = 1 \rangle$$

- and there is an epimorphism

$$\xi : \pi_1(T^\circ, Q_0) \to G$$

- define

$$c_j = \xi(\gamma_j)$$

- and set

$$n_j = o(c_j) \tag{1}$$

- $(0; n_1, \ldots, n_t)$ is called the signature of the G action

Group actions - 3

- $(c_1, \ldots, c_t)$ is called a *generating vector* for the action
- the generating vector satisfies

$$G = \langle c_1, \ldots, c_t \rangle \tag{2}$$

and

$$c_1 \cdots c_t = Id(G) \tag{3}$$

Proposition

The group G acts on a surface of genus

$$\sigma = 1 + \frac{|G|}{2}\left(t - 2 - \sum_{j=1}^{t} \frac{1}{n_j}\right)$$

if and only if there is a generating vector $(c_1, \ldots, c_t)$ satisfying equations 1, 2, and 3.

Actions of $\mathbb{Z}_p - 1$

- assume $G = \mathbb{Z}_p$
- the signature is $(p, \ldots, p)$ (t times) and the genus is

$$\sigma = \frac{(t-2)(p-1)}{2}$$

- a unique fixed point P_j lies over each Q_j
- the Eichler trace formula is easier to apply
- the characters are defined by:

$$\chi_j(c) = \zeta^{cj}, \ \zeta = \exp(2\pi i/p)$$

Actions of $\mathbb{Z}_p$ - 2

- the multiplicity matrix is $p \times p$

$$M = \left[\mu_q^j\right], \; j = 0, \ldots, p-1, \; q = 1, \ldots, p$$

- a generating vector is a set of integers satisfying

$$1 \le c_1, \ldots, c_t \le p-1$$

$$c_1 + \cdots + c_t = 0 \bmod p$$

- there are many of these, but we reduce this in the next section

Topological Equivalence

Definition

Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are topologically equivalent if there is an intertwining homeomorphism $h : S_1 \to S_2$ and an automorphism $\omega \in \mathrm{Aut}(G)$ such that

$$\epsilon_2(g) = h\epsilon_1(\omega(g))h^{-1}, \forall g \in G.$$

- topological equivalence converts the classification of actions into a discrete problem of classifying generating vectors
- for abelian groups we can arbitrarily permute elements of a generating vector with the same order
- for any group we may apply a group automorphism to all elements of a vector to get an equivalent action

Branch Locus of Moduli Space

Why is topological equivalence important? (Broad statements with little explanation here.)

- Topological equivalence classes correspond to conjugacy classes of finite subgroups of the mapping class groups.
- Topological equivalence classes correspond to "equisymmetric strata" in the branch locus of the moduli space, via the mapping class group action on Teichmüller space.
- An equisymmetric stratum corresponds to a parametric family of surfaces with the "same" automorphism group, usually G. The correspondence is not $1-1$ but is well understood.

Prime cyclic *n*-gonal strata

- In the *n*-gonal case, the parametric family is determined by the coordinates of $Q_1, \ldots, Q_t$ and has $t-3$ independent parameters.
- A prime cyclic group is a minimal action subgroup, so the corresponding stratum is maximal (Galois relations are satisfied).

$\mathcal{H}^q$ Equivalence

Definition

Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are $\mathcal{H}^q$ equivalent if the representations on $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$ are equivalent representations. Equivalently, the characters of G afforded by $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$ are equal.

The following Proposition may be proven using the Eichler trace formula.

Proposition

If two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are topologically equivalent then they are $\mathcal{H}^q$ equivalent.

Small examples

Example

For genus 4, p=5 and t=4 we get the smallest genus example. See Table 1.

Example

Some additional examples for small primes See Table 2.

Other Examples

Example

The number of conflated actions can be quite large. For example for $p = 11, t = 10, q = 1, \sigma = 40$ there are 854 actions with numerous conflations. In one case 31 different actions all have the same $\mathcal{H}^1$character. For $p = 19, t = 10, q = 1, \sigma = 63$ there are 9143 actions with a maximum of 116 conflated actions.

Example

See Table 3 for more examples on the the values of q for which conflation can occur.