Thomae type formulas for general cyclic covers of $\mathbb{CP}^1$

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Preliminaries

Assume : $\phi : X \mapsto \mathbb{CP}^1$ be a cyclic cover of the sphere of degree N given by the equation:

$$y^N = \prod_{i=}^{m} (x - \lambda_i)^{R_i}$$

such that $(R_i, N) = 1, \sum_{i=1}^{m} R_i = 0 modN$ Choose:

1. a base point z_0,
2. Normalized homology basis $a_1 ... a_g, b_1 ... b_g$
3. Normalized holomorphic differentials $v_1, ... v_g$
4. Non ramification $z_0 \in X$ and $z_0 \neq \infty$

These choices define the period τ of X and a standard mapping $u : X \mapsto J(X)$. Let K_{z_0} be the Riemann constant. i.e. a well defined point J(X) such $2K_{z_0} = \Delta$, $u(\Delta)$ Δ is the divisor of a differential.

Main result

Theorem

Let r be a total ramification of $\phi : X \mapsto \mathbb{CP}^1$. Select an integer vector $\beta = (\beta_1 ... \beta_m)$ such that:

1. $0 \leq \beta_i \leq N - 1$
2. *for* $0 \leq k \leq N - 1$, $\sum_{i=1}^{m} (\beta_i + kR_i) mod N = \frac{r}{2}$

Then:

1. $\theta \left[u(\sum_{i=1}^{m} \beta_i P_i) - K_{z_0} - u(\sum_{j=1}^{N} \infty_j) \right] (0, \tau) \neq 0$
2. *There is a complex number C not depending on τ such that:* $\theta \left[u(\sum_{i=1}^{m} \beta_i P_i) - K_{z_0} - u(\sum_{i=1}^{N} \infty_i) \right] (0, \tau) = C det(A) \prod_{i,j=1, i \neq j}^{m} (\lambda_i - \lambda_j)^{\gamma_{ij} + \beta_{ij}}$

Statement of the main result

$detA$ is a certain determinant of the $g \times g$ matrix of non normalized holomorphic differentials evaluated at $a_i, 1 \leq i \leq g$ and

$$\gamma_{ij} = \sum_{w=0}^{N-1} \{wR_i/N\} \{wR_j/N\},$$

$$\beta_{ij} = \sum_{k=0}^{N-1} \{(\beta_i + kR_i)/N\} \times \{(\beta_j + kR_j)/N\}$$

and $\{\alpha\}$ is the fractional part of α.

Example

Consider the case $N = 3$ and the equation:
$y^3 = \prod_{i=1}^{3n} (x - \lambda_i)$. Then $\beta_i = 0, 1, 2$. To select β_i satisfying the conditions of the theorem. Divide the set of points $\lambda_1 ... \lambda_{3n}$ into 3 equal sets: $\Lambda_0, \Lambda_1, \Lambda_3$. Then calculating β_{ij} we get the following corollary for this case:

Corrolary

For each partition of $3n$, $\Lambda = (\Lambda_0, \Lambda_1, \Lambda_2)$ *into equal parts (n elements in each) define the divisor:*
$D_\Lambda = u(\sum_{i \in \Lambda_j} jP_i - \sum_{i=1}^{3n} \infty_i)$. *Further for such a partition define the polynomial:* $p_\Lambda = \prod_{i,k \in \Lambda_j, i \neq k} (\lambda_i - \lambda_k)$ *Then*
$\theta^6 [D_\Lambda - K_{z_0}] (0, \tau) = C \times detA \times p_\Lambda^2 \times \prod_{i \neq j, j=1...3n} (\lambda_i - \lambda_j)$

Example N=3 continued

The polynomials p_Λ in the last example are of some interest. Using the representation theory of the Symmetric group we show:

Theorem

The natural action of S_{3n} on polynomials p_Λ and hence $\pm\theta^3\left[D_\Lambda - K_{z_0}\right](0,\tau)$ is an irreducible representation of S_{3n} of dimension:$\frac{(3n)!\times 2}{(n+2)!(n+1)!n!}$

Sketch of the proof - non vanishing of the divisors

We show the non vanishing theta functions on the divisors using the following criterion which follows from Riemann's theorem:

Lemma

Let D, $degD = g - 1$ be a non positive divisor such that $r(-D) = 0$ then $\theta\left(u(D) - K_{z_0}\right) \neq 0$. Where for divisor D, $r(-D) = dimH^0\left(X, \mathcal{O}\left(-D\right)\right)$

Thus it is enough to show that $r(-D) = 0$ where D is : $\sum_{i=1}^{m} \beta_i P_i - \sum_{j=1}^{N} \infty_j$ of degree $g - 1$. To compute the dimension of any divisor D observe defined on X note: $H^0\left(X, \mathcal{O}\left(-D\right)\right) = \bigoplus_{i=0}^{N-1} V_i y^i$ and V_i is a lift of functions on the $\mathbb{CP}^1$ to X. We calculate the dimension of V_i using Riemann Roch on $\mathbb{CP}^1$ and discover that $dimV_i = 0$. Hence $r(-D) = 0$ and $\theta\left(u(D) - K_{z_0}\right) \neq 0$.

Sketch of the proof - Variational method

The proof of the formula proceeds along the lines of Nakayashiki who showed the formula for the non singular case, i.e. for the case $y^N = \prod_{i=1}^{mN}(x - \lambda_i)$. The idea of Nakayashiki is to find these formulas using the "local-global" approach. More specifically Fay discovered formulas that connect certain analytical quantities on Riemann surface X using theta functions (Global computation). for the non singular case Nakayashiki expressed these quantities as rational function of the branch points λ_i. (local computation.) equating these expressions gives us the formulas we seek. We observed that these calculations can be carried in the more general setting of the curve $y^N = \prod_{i=}^{m}(x - \lambda_i)^{R_i}$. In what follows we explain partially the quantities and formulas involved to communicate the spirit of the technical computation.

Holomorphic quantities on X

The most basic analytical to build analytical quantities locally is the basis of holomorphic (non-normalized!) differentials. For the case of cyclic covers these are easy to construct. The following lemma is true:

Lemma

Let $s_l(z) = \prod_{i=1}^{m} (x - \lambda_i)^{\frac{\overline{lR_i}}{N}}.$, $0 \leq l \leq N-1$ *Then a basis for holomorphic differentials is given by:*

$$\frac{z^{j-1} dz}{s_l(z)},$$

where $j = 1 \ldots d(l), d(l) = Max \left(\sum_{i=1}^{m} \frac{\overline{lR_i}}{N} - 1, 0 \right).$

To verify the lemma calculate the order of vanishing of the differentials at each P_i and $\infty_i, 0 \leq i \leq N-1$.

Example of a local global approach- canonical bi holomorphic differential

We give an example of the local global approach concentrating on the canonical bi-holomorphic differential.

Definition

The canonical symmetric differential is a $\omega(x, y)$ *is a meromorphic one differential with respect to* $x, y \in C$*, having a unique pole of second order when* z *tends to* w *with a leading expansion coefficient of* 1*. Further for a canonical homology basis* $a_i, b_j, 1 \leq i, j \leq g$ *we have:*

$$\int_{a_i} \omega(x, y) = 0$$

for fixed y*.*

Example of a local global approach- canonical bi holomorphic differential continued

Definition

Let $P = (x, y) \in X$ be a non branch point with a local coordinate z. Define: $G_z(z) = \lim_{y \to x} \left[\omega(x, y) - \frac{dz(x)dz(y)}{(z(y)-z(x))^2} \right]$

Now taking the local coordinate $t = (z - \lambda_i)^{\frac{1}{N}}$ around the branch point λ_i we showed the following corollary:

Corrolary

The coefficient of $t^{N-2}dt^2$ in the expansion of $G_z(z)$ in $t = (z - \lambda_i)^{\frac{1}{N}}$ is: $-N \sum_{j=1, j \neq i}^{m} \frac{\gamma_{ij}}{\lambda_i - \lambda_j} - NlogdetA$, *where* $\gamma_{ij} = \sum_{h=0}^{N-1} \{hR_i/N\} \{hR_j/N\}$

And $detA$ is the matrix obtained by integrating the non normalized differentials defined above with respect to a_i.

Example of a local global approach- canonical bi holomorphic differential continued

On the other hand we can construct this object globally using theta functions and their derivatives evaluated at the divisors defined above. Once again we can calculate expansions using the local coordinate $(z-\lambda_i)^{\frac{1}{N}}$ around each branch point using global expressions. This isn't as complicated as it looks but requires patience and tenacity. Since we calculate the same object equating the coefficients of the power t^{N-2} produces the desired formula.

Conclusion

Here are some final thoughts to conclude this short talk. The pursuit of explicit formulas to describe values of theta functions at periods might present technical challenge which is worthwhile:

1. valuating at certain points and passing to the infinite series should give us fascinating identities generalizing the famous identities of $\eta(\tau)$ that is evaluated at certain special points (complex multiplication)
2. Produces dimensions of spaces of higher dimensional modular forms using representation theory of finite groups (see N$=$3 example above)
3. What about other fields? Inspecting the formulas shows that the RHS is well defined for any curve and any field. LHS? Algebraic theta functions?? something else?
4. Point counting ala Mestre !