Edge-transitive tessellations with non-negative Euler characteristic

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2-CELL EMBEDDING of a
Maps

- 2-CELL EMBEDDING of a
- 3-CONNECTED GRAPH on a
2-CELL EMBEDDING of a 3-CONNECTED GRAPH on a COMPACT CLOSED SURFACE
Maps

- 2-CELL EMBEDDING of a 3-CONNECTED GRAPH on a COMPACT CLOSED SURFACE

Automorphism → automorphism of the graph that extends to an auto-homoeomorphism of the surface.
2-CELL EMBEDDING of a 3-CONNECTED GRAPH on a COMPACT CLOSED SURFACE

- Automorphism $\rightarrow$ automorphism of the graph that extends to an auto-homomeorphism of the surface.
- Flag $\rightarrow$ triple of incident vertex, edge and cell (face)
Flag graph
Flag graph
Delaney-Dress symbol $\rightarrow$ flag graph / automorphism group
Delaney-Dress symbol → flag graph / automorphism group

regular
Delaney-Dress symbol → flag graph / automorphism group

regular

chiral
Edge-transitive maps have 1-, 2-, or 4-orbits under the automorphism group.
Edge-transitive maps have 1- 2- or 4-orbits under the automorphism group

Edge type $\langle p, q; s, t \rangle$
Edge-transitive maps have 1-2- or 4-orbits under the automorphism group

Edge type \( \langle p, q; s, t \rangle \)

- \( p, q \longrightarrow \) size of faces
Edge-transitive maps have 1-2- or 4-orbits under the automorphism group.

Edge type $\langle p, q; s, t \rangle$

- $p, q \rightarrow$ size of faces
- $s, t \rightarrow$ degrees of vertices
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- **Edge type** $\langle p, q; s, t \rangle$
  - $p, q \rightarrow$ size of faces
  - $s, t \rightarrow$ degrees of vertices

- **Edge homogeneous** $\rightarrow$ every edge has the same type
Edge-transitive maps have 1-2- or 4-orbits under the automorphism group.

- **Edge type** \( \langle p, q; s, t \rangle \)
  - \( p, q \rightarrow \) size of faces
  - \( s, t \rightarrow \) degrees of vertices

- **Edge homogeneous** \( \rightarrow \) every edge has the same type

- On planar tessellations
  - Edge-homogeneous \( \iff \) edge-transitive
Spherical tessellations

Tetrahedron $\langle 3, 3; 3, 3 \rangle$
Spherical tessellations

- Tetrahedron $\langle 3, 3; 3, 3 \rangle$
- Cube $\langle 4, 4; 3, 3 \rangle$
Spherical tessellations

- Tetrahedron $\langle 3, 3; 3, 3 \rangle$
- Cube $\langle 4, 4; 3, 3 \rangle$
- Octahedron $\langle 3, 3; 4, 4 \rangle$
Spherical tessellations

- Tetrahedron $\langle 3, 3; 3, 3 \rangle$
- Cube $\langle 4, 4; 3, 3 \rangle$
- Octahedron $\langle 3, 3; 4, 4 \rangle$
- Dodecahedron $\langle 5, 5; 3, 3 \rangle$
Spherical tessellations

- Tetrahedron \( \langle 3, 3; 3, 3 \rangle \)
- Cube \( \langle 4, 4; 3, 3 \rangle \)
- Octahedron \( \langle 3, 3; 4, 4 \rangle \)
- Dodecahedron \( \langle 5, 5; 3, 3 \rangle \)
- Icosahedron \( \langle 3, 3; 5, 5 \rangle \)
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- Icosidodecahedron $\langle 3, 5; 4, 4 \rangle$
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- Icosidodecahedron $\langle 3, 5; 4, 4 \rangle$
- Rhombic dodecahedron $\langle 4, 4; 3, 4 \rangle$
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- Icosidodecahedron $\langle 3, 5; 4, 4 \rangle$
- Rhombic dodecahedron $\langle 4, 4; 3, 4 \rangle$
- Rhombic triacontahedron $\langle 4, 4; 3, 5 \rangle$
Euclidean tessellations

<4,4;4,4>
Euclidean tessellations

\[ <4,4;4,4> \quad <3,3;6,6> \]
Euclidean tessellations

\(<4,4;4,4>\)  \(<3,3;6,6>\)  \(<6,6;3,3>\)
Euclidean tessellations

\[ <4,4;4,4> \]
\[ <3,3;6,6> \]
\[ <6,6;3,3> \]
\[ <3,6;4,4> \]
Euclidean tessellations

<4,4;4,4>  <3,3;6,6>  <6,6;3,3>

<3,6;4,4>  <4,4;3,6>
Hyperbolic tessellations

- \{3,7\}
- \{7,3\}
Admissible types

1. \( p = q, s = t \): regular (reflexible)
Admissible types

1 \( p = q, s = t \): regular (reflexible)
2.1 \( s = t, \text{ even, } p > q \), both odd: vertex-transitive
2.2 \( s = t, \text{ even, } p > q \), all even: vertex-transitive
2.3 \( s = t, \text{ even, } p > q \), one odd: vertex-transitive
1. $p = q, s = t$: regular (reflexible)

2.1 $s = t$, even, $p > q$, both odd: vertex-transitive

2.2 $s = t$, $p > q$, all even: vertex-transitive

2.3 $s > t$, even, $p = q$, one odd: vertex-transitive

3.1 $s > t$, $p = q$, all even: face-transitive

3.2 $p = q$, even, $s > t$, one odd: face-transitive

3.3 $p = q$, even, $s > t$, both odd: face-transitive
Admissible types

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3.1. $p > q$, even, $s > t$, all even: face-transitive

3.2. $p = q$, even, $s > t$, one odd: face-transitive

3.3. $p = q$, even, $s > t$, both odd: face-transitive

4. $p > q, s > t$, all even
Edge-transitive maps on compact surfaces
Edge-transitive maps on compact surfaces → quotients of an edge-transitive planar tessellation
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quotients of an edge-transitive planar tessellation

Quotients of the sphere
Edge-transitive maps on compact surfaces → quotients of an edge-transitive planar tessellation

Quotients of the sphere → Projective plane
Compact surfaces

- Edge-transitive maps on compact surfaces → quotients of an edge-transitive planar tessellation
- Quotients of the sphere → Projective plane
- Quotients of the Euclidean plane
Compact surfaces

- Edge-transitive maps on compact surfaces → quotients of an edge-transitive planar tessellation
- Quotients of the sphere → Projective plane
- Quotients of the Euclidean plane → Torus, Klein bottle
Projective plane
Projective plane

Hemi-cube $\langle 4, 4; 3, 3 \rangle$
Projective plane

- Hemi-cube $\langle 4, 4; 3, 3 \rangle$
- Hemi-octahedron $\langle 3, 3; 4, 4 \rangle$
Projective plane

- Hemi-cube $\langle 4, 4; 3, 3 \rangle$
- Hemi-octahedron $\langle 3, 3; 4, 4 \rangle$
- Hemi-dodecahedron $\langle 5, 5; 3, 3 \rangle$
Projective plane

- Hemi-cube $\langle 4, 4; 3, 3 \rangle$
- Hemi-octahedron $\langle 3, 3; 4, 4 \rangle$
- Hemi-dodecahedron $\langle 5, 5; 3, 3 \rangle$
- Hemi-icosahedron $\langle 3, 3; 5, 5 \rangle$
Projective plane

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- Hemi-dodecahedron $\langle 5, 5; 3, 3 \rangle$
- Hemi-icosahedron $\langle 3, 3; 5, 5 \rangle$
- Hemi-cuboctahedron $\langle 3, 4; 4, 4 \rangle$
• Hemi-cube $\langle 4, 4; 3, 3 \rangle$
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• Hemi-icosahedron $\langle 3, 3; 5, 5 \rangle$
• Hemi-cuboctahedron $\langle 3, 4; 4, 4 \rangle$
• Hemi-icosidodecahedron $\langle 3, 5; 4, 4 \rangle$
Projective plane

- Hemi-cube $⟨4, 4; 3, 3⟩$
- Hemi-octahedron $⟨3, 3; 4, 4⟩$
- Hemi-dodecahedron $⟨5, 5; 3, 3⟩$
- Hemi-icosahedron $⟨3, 3; 5, 5⟩$
- Hemi-cuboctahedron $⟨3, 4; 4, 4⟩$
- Hemi-icosidodecahedron $⟨3, 5; 4, 4⟩$
- Hemi-rhombic dodecahedron $⟨4, 4; 3, 4⟩$
Projective plane

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Admissible types

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4. $p > q, s > t$, all even
Every edge-transitive tessellation has one of the previous admissible types.
Admissible types

- Every edge-transitive tessellation has one of the previous admissible types

- There are infinitely many maps on compact closed surfaces with any given hyperbolic type
Higher genus

▶ Alen’s Orbanić’s PhD
Higher genus

- Alen’s Orbanić’s PhD

Edge type
Higher genus

- Alen’s Orbanić’s PhD
- Edge type
- Delaney-Dress graph