Genus 2 Belyĭ surfaces with a unicellular uniform dessin

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Definition

A **dessin d'enfant** *is a bipartite graph embedded into an oriented compact surface S and dividing it into topological discs, called the faces of the dessin.*

We say that $\mathcal{D}$ is **uniform** *when all of its white vertices (resp. black vertices, faces) have the same valency p (resp. the same valency q, the same valency $2r$), and (p, q, r) is said to be the type of $\mathcal{D}$.*

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Such a dessin corresponds to the inclusion of a (torsion free) group K uniformizing S inside a triangle group $\Delta = \Delta(p, q, r)$.

The corresponding Belyĭ function can be seen as the projection $\beta : \mathbb{D}/K \longrightarrow \mathbb{D}/\Delta$.

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Let K be the Fuchsian group uniformizing S:

1. There exists some $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$.
2. There exists some $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that $K < \Delta$ and $\alpha^{-1}K\alpha < \Delta$.

Multiple dessins lying on the same surface

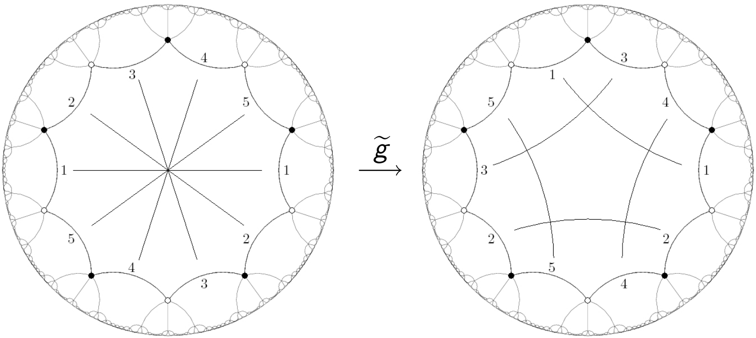

Figure: Two uniform dessins on the curve $y^2 = x^5 - 1$. They correspond to different inclusions $K < \Delta(5,5,5)$ and $\widetilde{g}K\widetilde{g}^{-1} < \Delta(5,5,5)$.

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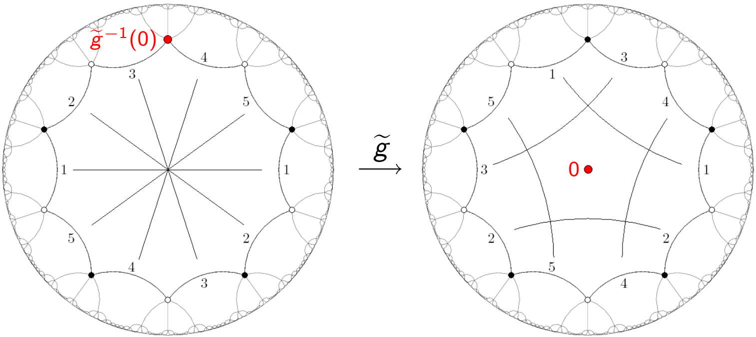

Figure: A lift of the isomorphism g^{-1} identifies the center of the second dessin in the fundamental polygon of the first.

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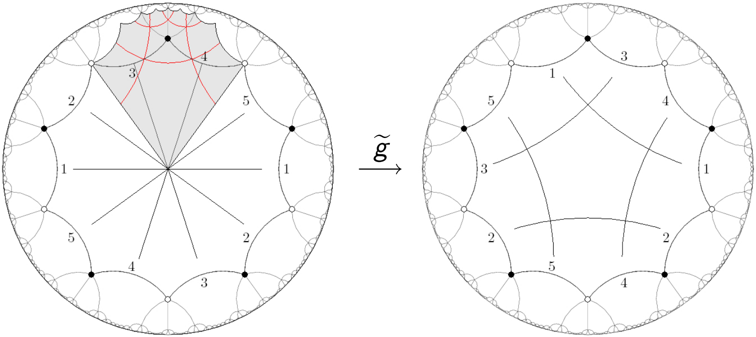
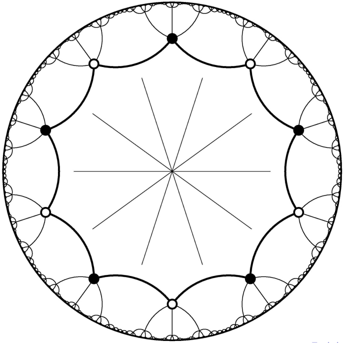


Figure: The side-pairings of the fundamental polygon of K around $\widetilde{g}^{-1}(0)$ coincide with the ones of the second dessin.

Admissible distances

Let $K < \Delta(p,q,r)$ correspond to a $(p,q,r)-$dessin on $\mathbb{D}/K$. Consider the tessellation of the disc in $2r-$gons associated to $\Delta(p,q,r)$.

All the transformations in K translate the origin to another polygon center (point of order r) in the disc.



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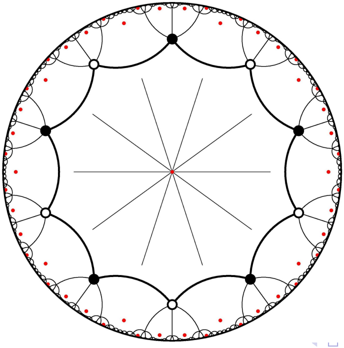

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Definition

We define the set of **admissible distances** *of type* (p, q, r)*:*

$$d(p, q, r) = \{d_1 < d_2 < d_3 < \dots \}$$

as the set of (hyperbolic) distances between any two $2r-$*gon centers as above.*

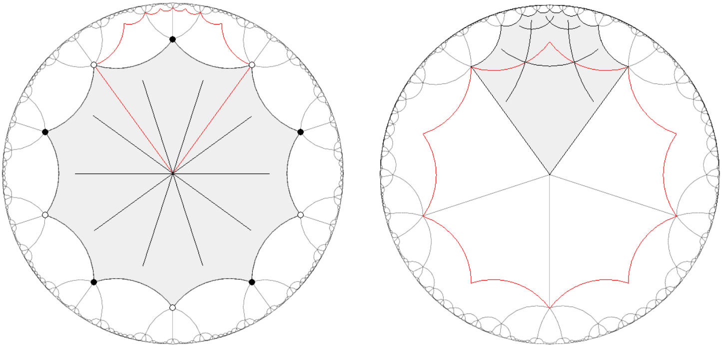

Figure: The points in $\mathbb{D}$ corresponding to centers of dessins in $\mathbb{D}/K$ are translated admissible distances by the elements of K.

A criterion to find uniform dessins

Lemma

Let $\mathcal{D}$ be a dessin given by the inclusion $K < \Delta(p, q, r)$, and let $\mathcal{D}'$ be another dessin of the same type in the same surface. If $z \in \mathbb{D}$ corresponds to a face center of $\mathcal{D}'$, then for every $g \in K$ the hyperbolic distance $\rho(z, g(z))$ is an admissible distance.

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Any point of $\mathbb{D}$ moved an admissible distance by every transformation in K will be called an **admissible point**.

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How is the set of points of $\mathbb{D}$ which are translated a fixed distance by a given $g \in K$?

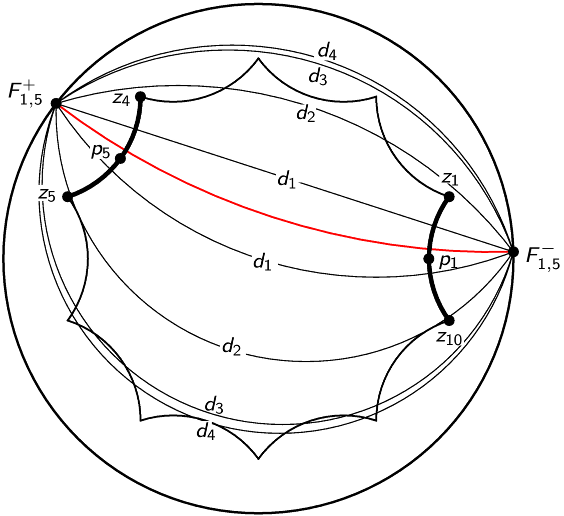

Figure: Some of the admissible arcs in the case $(p, q, r) = (5, 5, 5)$.

The main result

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This statement is known to be false in general.

- There is a genus 2 surface containing two (not unicellular) uniform dessins of type $(2, 3, 9)$;
- Klein's surface of genus 3 contains two (not unicellular) uniform dessins of type $(2, 3, 7)$.

(E.Girondo, J.Wolfart, D.T.)

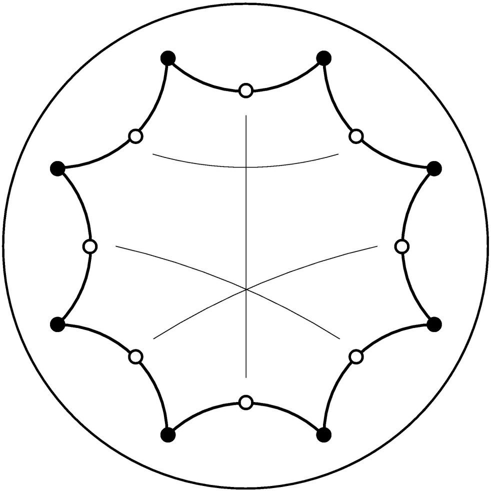
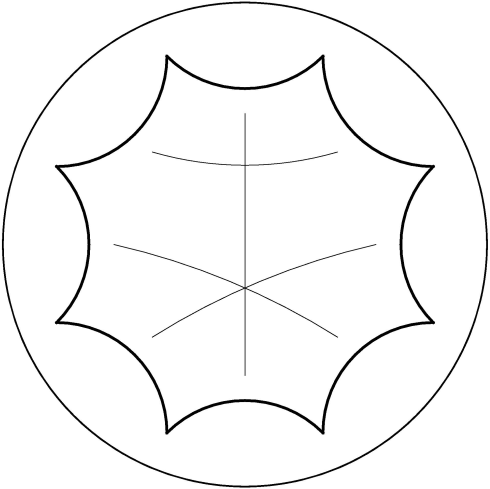
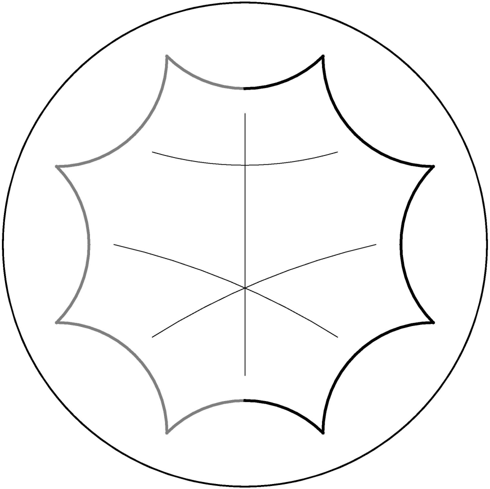
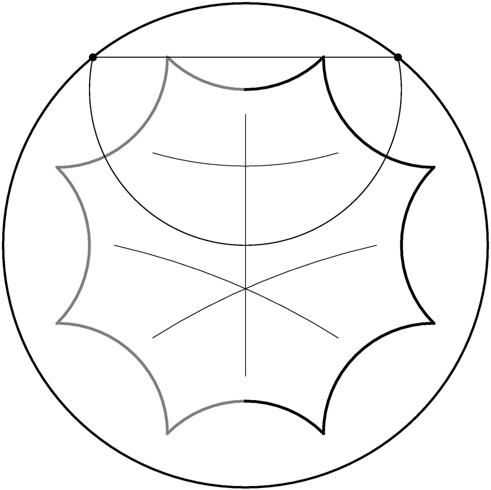
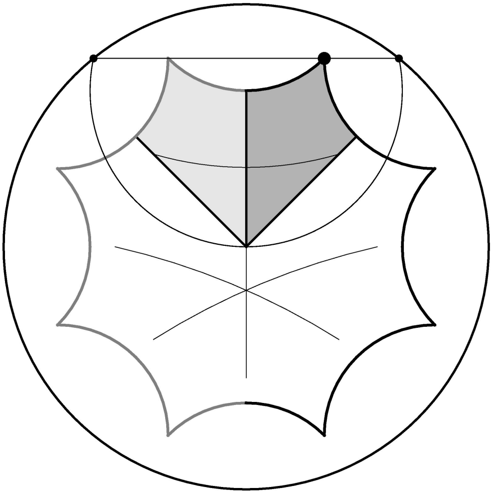


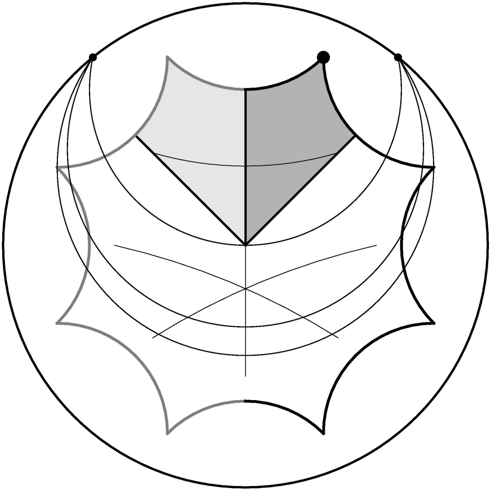
Figure: Our dessin $\mathcal{D}$ given by an inclusion $K < \Delta(2, 8, 8)$.

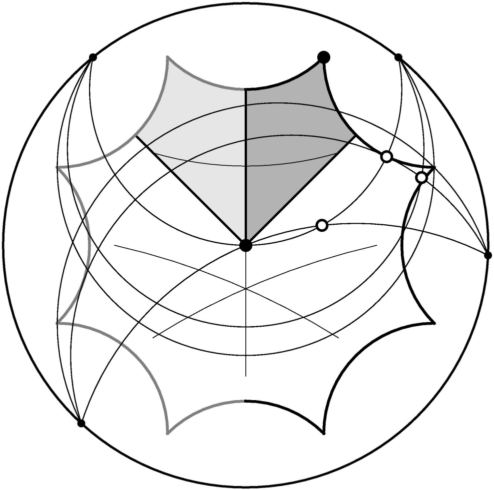


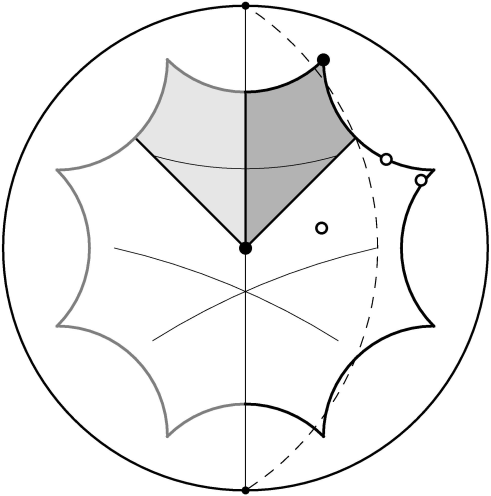


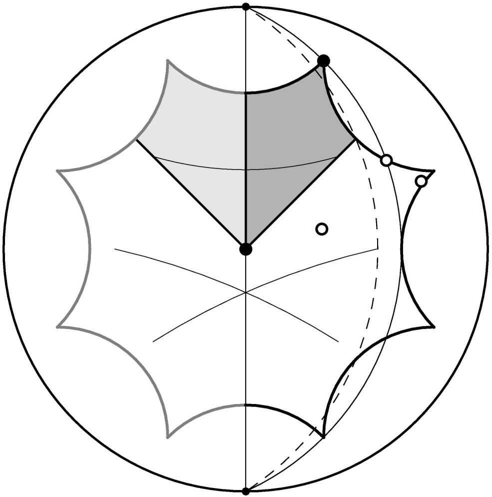


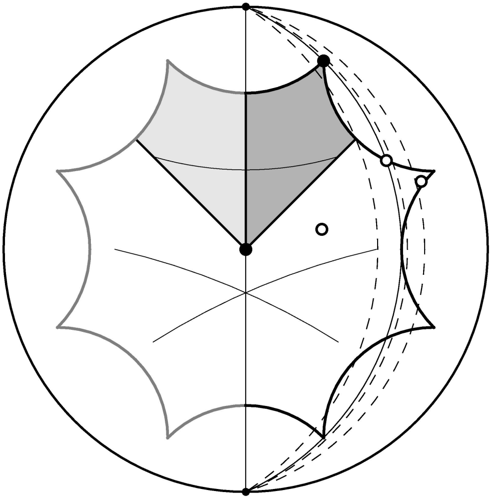


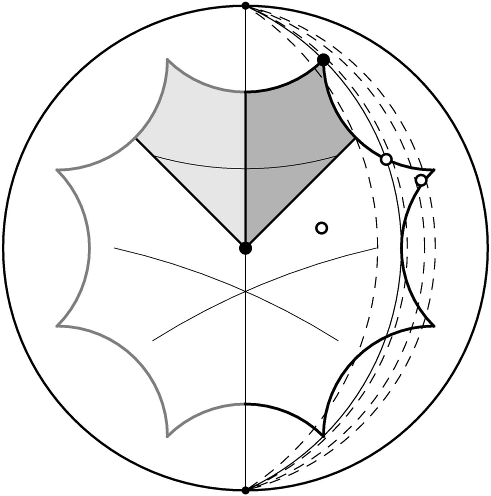


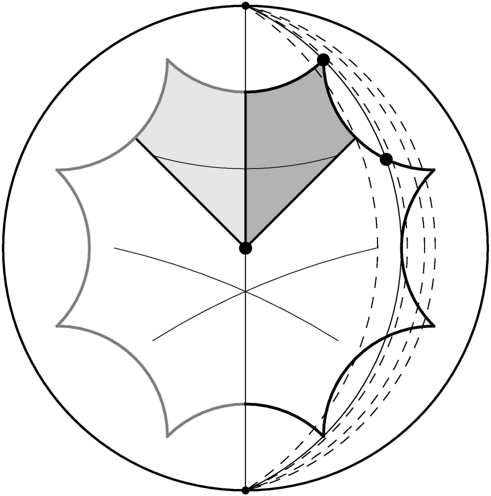


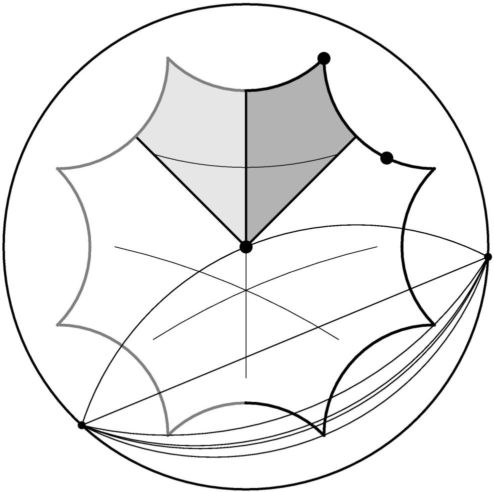


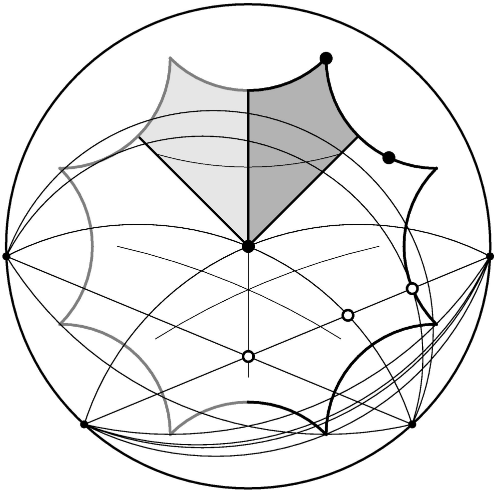


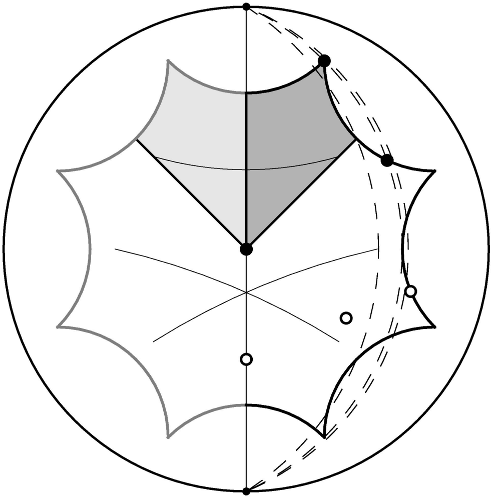


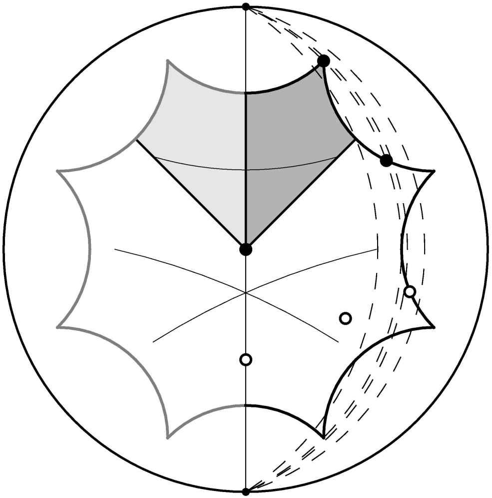


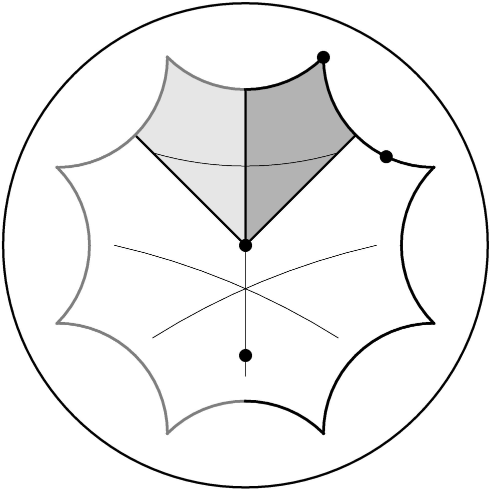


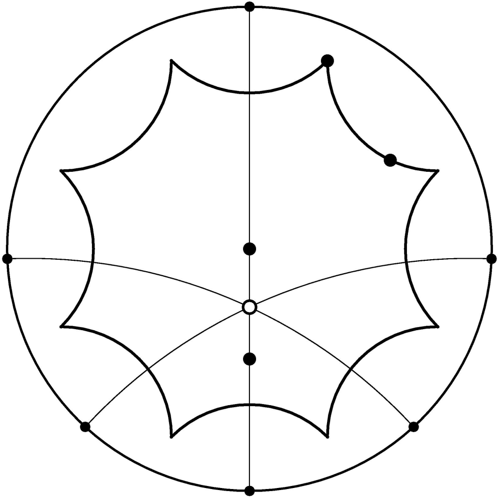












Thank you!!!